



MATHEMATICAL FINITE ELEMENT MODEL FOR GENERAL ANALYSIS OF DOUBLE CURVED SHELL ACCORDING TO STRAIN BASED APPROACH

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Abstract: A new mathematical finite element model suitable for the general bending analysis of double curved shell structures depending on the strain based approach has been derived. The element is simple and contains only the essential degrees of freedom. The element has the advantage over the other available double curved shell elements. The improvement obtained is due to the fact that all the displacement fields of the present element satisfy the exact representation of rigid body modes of displacements then the shape function error due to rigid body modes becomes zero. Also, the present element satisfies the full geometry of the double curved shell due to this point discretization error becomes zero. Finally, the error due to strain mode becomes very small because the present element satisfies the compatibility equations of strains and the 19 coefficients of strain mode derived exactly from partial differential equations of strains. The numerical solution of several problems by using the present element proved to be powerful in the structural analysis of double curved shells, such as cylindrical shells. Its results are better than the solution of other elements and packages with respect to analytical solution.

Keywords: shell, finite element, structure, rigid body mode, strain mode, and stress field

نموذج رياضي لعنصر محدد ثنائي التقوس للتحليل العام للمنشآت القشرية اعتماداً على مبدأ الانفعال

الخلاصة: تم اشتقاق عنصر قشري ثنائي الانحناء محدد يعتمد على طريقة الأنفعال حيث يظم درجات الطلاقة الرئيسية. العنصر يحقق مواصفات الحركة للجسم الصلب بشكل تام (exact rigid body mode)، كما يحقق كامل الخواص للشكل الهندسي للعنصر القشري (full geometry of double curved shell)، إضافة إلى ذلك فإن العنصر يحقق شروط التوافق لمعادلات الأنفعال (compatibility equation of strain). يعتبر هذا العنصر ملائم لتحليل الأنحناي العام للمنشآت القشرية المختلفة وهو أفضل من العناصر السابقة لتحليل المنشآت الأنفة الذكر حيث أن الأخطاء التي تظهر في العناصر المحددة السابقة مثل أخطاء التقسيم (discretization error)، أخطاء الدوال الشكلية (shape function error) تختفي في هذا العنصر نتيجة للمواصفات التي يتمتع بها. تم استخدام العنصر الحالي في تحليل عدة مسائل للقشور الأسطوانية. إن النواتج العددية للتحليل باستخدام العنصر الحالي تم مقارنتها مع نواتج أخرى لتحليلات رياضية، وتحليلات عددية لنتائج الباحثين الآخرين، حيث أظهر العنصر الجديد نتائج جيدة جداً ويحقق اقتراب سريع من نتائج التحليل الرياضي والعددي للباحثين السابقين وبعده قليل من العناصر مقارنة مع عدد العناصر المستخدمة من قبل الباحثين السابقين. إن نسبة الخطأ في الحل لأغلب المسائل كانت أقل من 0.1%. التحليل العددي لعدد من المسائل باستخدام العنصر الحالي تبين وتبرهن أن العنصر الحالي ممتاز وكفؤ في تحليل عدة أنواع من المنشآت بشكل أفضل من العناصر المحددة لباحثين آخرين.

1. Introduction

An alternative approach for the analysis of shells by the finite element method was to develop curved elements that would be able to represent a particular shell surface geometrically. Jones and Storm [1] and Strickland et al [2] have modified the method for a shell of revolution. They used curved meridional elements rather than conical segments. The limited number of comparison studies carried out indicates that curved elements lead to considerably better results for the stresses. In fact, Navaratna [3] found that the stress discontinuity, arising when using different size elements in the same mesh, essentially disappeared when curved rather than conical segments were used.

2. Literature review

In order to obtain more accurate solutions for shells, a stiffness matrix for a curved element, which provides the coupling between the bending and the membrane actions, is needed. This coupling is usually done by, including the curvature terms in the strain-displacement relationships. A number of curved rectangular elements have been developed using different displacement functions to express the in plane and out of plane displacements within the elements. The simplest of these was that of Conner and Brebbia [4]. This element is based on shallow shell theory. The in plane displacements (u) and (v) were expressed by the usual bilinear displacement functions in terms of the coordinate variables (X) and (Y):

$$u = a_1 + a_2x + a_3y + a_4xy \quad (1)$$

$$v = a_5 + a_6x + a_7y + a_8xy$$

and the out of plane displacement (w) by the well-known non-conforming (12 terms) plate bending displacement function of Zienkiewicz and Cheung [5]:

$$w = a_9 + a_{10}x + a_{11}y + a_{12}x^2 + a_{13}xy + a_{14}y^2 + a_{15}x^3 + a_{16}x^2y + a_{17}xy^2 + a_{18}y^3 + a_{19}x^3y + a_{20}xy^3 \quad (2)$$

The element possesses three principal curvatures (two direct and one twisting). Five degrees of freedom were considered at each node, namely, u, v, w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. This leads to a (20X20) element stiffness matrix.

Conner and Brebbia tested this element to analyze a clamped hyperbolic paraboloid shell and obtained satisfactory results using a 12X12 mesh of elements. However, they reported that the coupling due to the curvature in the out of plane strain equations was neglected and that the expressions for (u) and (v) did not contain all the rigid body modes of displacements. Bogner et al [6] expressed the out of plane displacement (w) in terms of a 16-terms cubic polynomial, in which the additional terms to (w) were:

$$a_{21}x^2y^2 + a_{22}x^3y^2 + a_{23}x^2y^3 + a_{24}x^3y^3 \quad (3)$$

and the in plane displacements (u) and (v) as above, and obtained a (24X24) element stiffness matrix for a cylindrical shell. The additional nodal degree of freedom was $\frac{\partial w}{\partial x \partial y}$. They then proceeded to develop a (48X48) stiffness matrix in which (u), (v), and (w) were all expressed by the same 16-terms polynomial used for (w) as above. In this

element, the additional degrees of freedom were $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, and $\frac{\partial^2 v}{\partial x \partial y}$. This latter element showed superior convergence characteristics over the other available elements apparently due to the use of higher order in plane displacement functions. It was also claimed that all the rigid body modes of displacements (essential for a shell element) were implicitly included. This, however, can only be true in the limit where the elements are very small since the exact rigid body displacements of a shell element cannot be expressed in terms of polynomial functions. Cantin and Clough [7] pointed out that none of the previously considered polynomial forms for (u), (v), and (w) allowed all six rigid body displacements of the shell element. They removed this restriction by including terms containing trigonometric functions. A more convenient set of nodal degrees of freedom than those for previously mentioned elements was used; u, v, w. The example shown for a cylindrical shell segment under gravity loads and a pinched cylindrical shell indicate that the inclusion of rigid body displacement terms permits the attainment of better results with relatively coarse mesh.

A number of triangular elements were also developed. Bonnes et al [8] used complete cubic polynomials for the displacements u, v, w. Nine degrees of freedom were specified at each corner and three at the mid side nodes thus making a (36X36) element stiffness matrix. This high order element, however, was shown by Sabir and Lock [9] to be unsatisfactory when tested in the analysis of hyperbolic paraboloid shell problems. The work on the development of high order elements was continued and one of the most successful elements was developed, by Cowper, et al. [10]. It is a conforming shallow shell element of arbitrary triangular shape. They pointed out that using shallow shell approximation leads to significant simplification that all necessary mathematical manipulations may be carried out in the base reference plane and it is sufficient to assume constant curvature over the element. The displacement function for the out of plane displacement (w) of the shell was taken as a quintic polynomial (21 terms) in the two cartesian coordinates in the base plane.

Three constraints are placed on the polynomial to ensure that normal derivative vary cubically along edges. The generalized coordinates (degrees of freedom) were, at each vertex: w and its first and second derivatives. The in plane displacements u and v were each expressed as cubic polynomial (10 terms) and the generalized coordinates were taken to be u, v and there are first derivatives at each vertex plus u, v at the centroid. However, these two centroidal displacements were condensed out of the final stiffness matrix, hence, the final element has 36 degrees of freedom and is completely

conforming to smooth shells. They tested this element on three shell problems, which show its superiority over previous developments.

Dawe [11] employed the same approach and used an even higher order element. His element is a curved triangular one and has a total of 54 degrees of freedom, in which, each of the three displacements u , v , w was independently represented by a quintic polynomial.

Yang [12] developed a shell element having rectangular projections on a plane with three constant radii of curvature (two direct and one twisting). The displacement parameters were approximated with one-dimensional, first-order, Hermite interpolation formulae. The element stiffness matrix was generated with the help of minimum potential energy. Eigenvalue analysis was performed on the element stiffness matrix yielded six nearly zero values corresponding to six independent rigid body modes. Three numerical examples including a cylindrical shell, a translational shell with two constant radii of curvature and clamped hyperbolic paraboloid shell were solved. The degree of accuracy for deflection profile at centerline remained within 1% with only 5X5 mesh size against the 12X12 mesh of Conner and Brebbia [4].

Bhimaraddi, et al. [13] used a modified isoparametric quadrilateral element with a 64 degree of freedom for the analysis of shells of revolution. The element is suitable for the analysis of shells of revolution subjected to any general class of loading (axisymmetric or asymmetric). This element was used in the analysis of a pinched-cylindrical shell, conical and hyperboloidal shells subjected to uniformly distributed edge loads, free vibrations of fixed-free circular cylindrical shell and laminated shells of revolution. The numerical results showed good convergence.

Most of the successful curved finite elements available to analyze shell problems are high order elements. However, there are some shortcomings about these elements. The high order elements not only lead to a considerable increase in the total number of unknowns to be solved but also lead to a much wider band width of the overall structural matrix. And, since the solution time is proportional to the number of arithmetical operations [8], then the computing effort or the execution time for high order elements becomes excessive and expensive. Also, the additional internal degrees of freedom are not associated with physical corresponding generalized forces and a problem arises whether these degrees of freedom need to be made continuous at the node.

Meanwhile, Sabir et al [14-21] has used a different approach for the development of curved elements. The method is based on the development of displacement functions, which satisfy the exact representation of strain-free rigid body modes of movements and on assumed independent strain, rather than displacement functions insofar as it is allowed by the elasticity compatibility equations. The resulting various components of the displacements are not independent, as in the usual displacement approach, but are linked. This linking is present in the exact terms representing rigid body modes and the approximate terms, within the context of the finite element method, representing the straining of the element. Another feature of the strain based approach is that the method allows the in plane components of the displacements to be represented by higher order terms than the out of plane components without increasing the number of degrees of

freedom beyond the essential external degrees of freedom. This is of particular interest since the improvement obtained by the high order element is mainly due to the representation of the in plane displacements by higher order polynomial terms.

Strain-based elements for arches deforming in the plane containing the curvature [14] and out of plane of curvature [14] as well as for cylindrical shell [16] were first developed. These elements, while possessing only geometric external degrees of freedom, were found to yield more accurate results and the solutions were obtained more rapidly than those for other more complex elements. This approach was further extended to develop a general quadrilateral cylindrical element [16] and was used [17] to investigate the problem of stress concentrations in cylinders having circular and elliptical holes, and also to obtain a solution to the problem of normally intersecting cylinder [17].

Moreover, Sabir and Ramadhani [18] developed an even simpler curved element for general shell analysis. The element is rectangular in plane and has only the essential five external nodal degrees of freedom at each of the four corner nodes, namely, the three displacements u , v , w and the two first partial derivatives of w with respect to the two cartesian axes. The simplicity of the element is due to the use of shallow, instead of deep, formulations. The element was tested by applying it to the analysis of cylindrical [18], as well as, spherical [19] shells and the results show a high degree of accuracy and can converge to the correct solution with relatively coarse meshes.

A curved strain based conical shell finite element suitable for the general bending analysis of conical shells has been developed by El-Eris[22]. The element is simple and contains only the essential degrees of freedom. The test problems carried out show that results of acceptable degree of accuracy can be obtained when few elements are used.

Mehdi, H. A.[23], developed an element suitable for the general bending analysis of conical shells depend on based strain approach. The element's displacement functions are developed in such a way that the six rigid body modes of displacements are exactly represented and the straining of the element is exactly represented by suitable polynomial expressions due to exact derivation from strain equations after satisfying the compatibility of assumed strain polynomial before the derivation. The element is simple and it possesses all the requirements for less computational work. The element is shown to provide satisfactory solutions for a range of conical shell bending problems.

3. Derivation of Double Curved Shell Element Based on Simple Independent Strain Functions

3.1. Theoretical Consideration

The thin shell theory, used to analyze the thin shell structures, provides a direct relationship between stresses and strains by which the equilibrium of the stress resultants and stress couples are related to the strains of compatibility equations, Flugge(24). For the general problem in three dimensions, there are six equilibrium conditions as shown in Figure (1), Finally, the strain equations of double curved shell element are as follows Flugge[24]:

$$\left. \begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{w}{R_x} \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{R_y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + 2 \frac{w}{R_{xy}} \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} \chi_x &= -\frac{\partial^2 w}{\partial x^2} \\ \chi_y &= -\frac{\partial^2 w}{\partial y^2} \\ \chi_{xy} &= -\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \quad (5)$$

where: ϵ_x , ϵ_y and γ_{xy} are the middle surface in-plane meridional, circumferential and shear strain, respectively. χ_x , χ_y and χ_{xy} are the middle surface changes in the curvature of (x) and (y), and the twisting curvature, respectively. If $Z = f(x, y)$ is the equation of the middle surface, then

$$\left. \begin{aligned} \frac{1}{R_x} &= -\frac{\partial^2 z}{\partial x^2} \\ \frac{1}{R_y} &= -\frac{\partial^2 z}{\partial y^2} \\ \frac{1}{R_{xy}} &= -\frac{\partial^2 z}{\partial x \partial y} \end{aligned} \right\} \quad (6)$$

Where R_x , R_y , and R_{xy} are the principle surface radii of curvature in the direction of x and y and the twist radius of curvature, respectively, of the unstrained element.

The above six components of strains cannot be considered independent as they are in terms of the three displacements u, v, w and hence the strains must satisfy three additional equations called the compatibility equations. These equations are obtained by eliminating the three displacements from equations (4), and (5). The final results of compatibility equations are as follows:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} + \frac{\chi_x}{R_x} + \frac{\chi_y}{R_y} - \frac{\chi_{xy}}{R_{xy}} = 0 \quad (7a)$$

$$\frac{\partial \chi_{xy}}{\partial x} - 2 \frac{\partial \chi_x}{\partial y} = 0 \quad (7b)$$

$$\frac{\partial \chi_{xy}}{\partial y} - 2 \frac{\partial \chi_y}{\partial x} = 0 \quad (7c)$$

3.2. Displacement Functions

It is assumed that the double curved shell element possesses only five external nodal degrees of freedom, namely, u , v , w , θ_x and θ_y . Proceeding as with the usual strain-based approach, the first major component of the displacement function is due to (strain-free) rigid body modes of displacement and can be obtained by equating all the components of strains, equations (4), and (5), to zero and integrating the resulting partial differential equations becomes:

$$u_R = a_1 \left(\frac{x}{R_x} + \frac{y}{R_{xy}} \right) + a_2 \left(\frac{x^2}{2R_x} - \frac{y^2}{2R_y} \right) + a_3 \left(\frac{xy}{R_x} + \frac{y^2}{R_{xy}} \right) + a_4 + a_6 y \quad (8a)$$

$$v_R = a_1 \left(\frac{y}{R_y} + \frac{x}{R_{xy}} \right) + a_2 \left(\frac{xy}{R_y} + \frac{x^2}{R_{xy}} \right) + a_3 \left(\frac{y^2}{2R_y} - \frac{x^2}{2R_x} \right) + a_5 - a_6 x \quad (8b)$$

$$w_R = -a_1 - a_2 x - a_3 y \quad (8c)$$

In these equations u_R , v_R , and w_R are the rigid body components of the displacement fields u , v and w , respectively, and are expressed in terms of the six independent constants (a_1 --- a_6). The second major component of the displacement function is due to straining of the element. If the element is to have twenty-five degrees of freedom (five at each node) then the strains of the element must be associated with nineteen additional constants (a_7 --- a_{25}). Assuming strain polynomial functions of (a_7 --- a_{25}) constants.

$$\varepsilon_x = a_7 + a_8 y \quad (9a)$$

$$\varepsilon_y = a_9 + a_{10} x \quad (9b)$$

$$\gamma_{xy} = a_{11} \quad (9c)$$

$$\chi_x = a_{12} + a_{13} x + a_{14} y + a_{15} xy + a_{21} x^2 + a_{22} y^2 \quad (9d)$$

$$\chi_y = a_{16} + a_{17} x + a_{18} y + a_{19} xy + a_{23} x^2 + a_{24} y^2 \quad (9e)$$

$$\chi_{xy} = a_{20} + a_{25}xy \quad (9f)$$

Checking the above polynomials of strain for compatibility equations of double curved shell (7a, 7b, and 7c). Finally, the assumed strain functions of double curved shell element which satisfy the requirement of compatibility equations become:

$$\varepsilon_x = a_7 + a_8y - \left[a_{16} \frac{y}{2} + a_{17} \frac{xy^2}{2} + a_{18} \frac{y^3}{6} + a_{19} \frac{xy^3}{6} + a_{23} \frac{x^2y^2}{2} + a_{24} \frac{y^4}{12} \right] \frac{1}{R_x} \quad (10a)$$

$$\varepsilon_y = a_9 + a_{10}x - \left[a_{12} \frac{x^2}{2} + a_{13} \frac{x^3}{6} + a_{14} \frac{x^2y}{2} + a_{15} \frac{x^3y}{6} + a_{21} \frac{x^4}{12} + a_{22} \frac{y^2x^2}{2} \right] \frac{1}{R_y} \quad (10b)$$

$$\gamma_{xy} = a_{11} - \left[\begin{array}{l} a_{14}x^2y + a_{15} \frac{x^3y}{3} + a_{17}xy^2 + a_{19} \frac{xy^3}{3} + a_{20}xy \\ + a_{22} \frac{y^2x^2}{2} + a_{23} \frac{x^2y^2}{2} \end{array} \right] \frac{1}{R_{xy}} \quad (10c)$$

$$\chi_x = a_{12} + a_{13}x + a_{14}y + a_{15}xy + a_{21}x^2 + a_{22}y^2 \quad (10d)$$

$$\chi_y = a_{16} + a_{17}x + a_{18}y + a_{19}xy + a_{23}x^2 + a_{24}y^2 \quad (10e)$$

$$\chi_{xy} = a_{20} + a_{25}xy + [2a_{14}x + a_{15}x^2 + 2a_{17}y + a_{19}y^2 + 2a_{22}x + 2a_{23}y] \quad (10f)$$

The above six equations 10a, b, c, d, e and f are integrated in the same procedure that was used to derive the rigid body modes of general double curved shell element, then the final polynomial function of strain mode becomes:

$$\begin{aligned} u_s = & a_7x + a_8xy - a_{10} \frac{y^2}{2} + a_{11} \frac{y}{2} + a_{12} \frac{x^3}{6R_x} + a_{13} \frac{x^4}{24R_x} + a_{14} \frac{x^3y}{6R_x} + a_{15} \frac{x^4y}{24R_x} \\ & a_{16} \frac{y^3}{3R_{xy}} - a_{17} \frac{y^4}{24R_y} + a_{18} \frac{y^4}{12R_{xy}} - a_{19} \frac{y^5}{120R_y} + a_{20} \left(\frac{x^2y}{4R_x} - \frac{y^3}{12R_y} \right) \\ & a_{21} \frac{x^5}{120R_x} + a_{22} \frac{y^2x^3}{6R_x} - a_{23} \frac{y^5}{120R_y} + a_{24} \frac{y^5}{60R_{xy}} + a_{25} \left(\frac{x^3y^2}{24R_x} - \frac{y^5}{240R_y} \right) \end{aligned} \quad (11a)$$

$$\begin{aligned}
v_S = & -a_8 \frac{x^2}{2} + a_9 y + a_{10} xy + a_{11} \frac{x}{2} + a_{12} \frac{x^3}{3R_{xy}} + a_{13} \frac{x^4}{12R_{xy}} - a_{14} \frac{x^4}{24R_x} - a_{15} \frac{x^5}{120R_x} \\
& + a_{16} \frac{y^3}{6R_y} + a_{17} \frac{xy^3}{6R_y} + a_{18} \frac{y^4}{24R_y} + a_{19} \frac{xy^4}{24R_y} + a_{20} \left(\frac{xy^2}{4R_y} - \frac{x^3}{12R_x} \right) \\
& a_{21} \frac{x^5}{60R_{xy}} - a_{22} \frac{x^5}{120R_x} + a_{23} \frac{x^2 y^3}{6R_y} + a_{24} \frac{y^5}{120R_y} + a_{25} \left(\frac{x^2 y^3}{24R_y} - \frac{x^5}{240R_x} \right)
\end{aligned} \quad (11b)$$

$$\begin{aligned}
w_S = & -a_{12} \frac{x^2}{2} - a_{13} \frac{x^3}{6} - a_{14} \frac{x^2 y}{2} - a_{15} \frac{x^3 y}{6} - a_{16} \frac{y^2}{2} - a_{17} \frac{xy^2}{2} - a_{18} \frac{y^3}{6} \\
& - a_{19} \frac{xy^3}{6} - a_{20} \frac{xy}{2} - a_{21} \frac{x^4}{24} - a_{22} \frac{x^2 y^2}{2} - a_{23} \frac{x^2 y^2}{2} - a_{24} \frac{y^4}{24} - a_{25} \frac{x^2 y^2}{24}
\end{aligned} \quad (11c)$$

The complete displacement field functions for the double curved shell element are, thus, the sum of the corresponding terms for u, v and w from equations (8) to (11) respectively.

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_r + \mathbf{u}_s \\
\mathbf{v} &= \mathbf{v}_r + \mathbf{v}_s \\
\mathbf{w} &= \mathbf{w}_r + \mathbf{w}_s \\
\theta_x &= \frac{\partial w}{\partial y} \quad \text{and} \quad \theta_y = -\frac{\partial w}{\partial x}
\end{aligned} \quad (12)$$

3.3. The Element Stiffness Matrix

Figure (1) shows that the origin of the meridional coordinate of the element (x) is located at the center of the double curved shell. Consequently, the origin of the circumferential coordinate (y) is located at the center of the element. Since, the calculation of the stiffness matrix is carried out explicitly; this choice of the origin will simplify the task of integration, thus:

$$\left[k^e \right] = \left[A^{-1} \right]^T \left[\int_{y_1}^{y_2} \int_{x_1}^{x_2} [B]^T [D] [B] dx dy \right] [A]^{-1} \quad (13)$$

where [A], [B] and [D] are the transformation, strain and rigidity matrices of the element, respectively. The (25X25) element stiffness matrix [ke], can now be calculated using the displacement functions (12) and the strain displacement relationships (10). On the other hand, to keep the storage memory small the stiffness matrix is condensates from (25x25) to (20x20) by removing the influence of the central point (node 5) to the four corner points (nodes 1, 2, 3, and 4) as follows:

$$\left[K^n_{20 \times 20} \right] \left\{ \delta^n_{20 \times 1} \right\} = \left\{ P^n_{20 \times 1} \right\} \quad (14)$$

where:

$$\begin{aligned} [K^n_{20 \times 20}] &= [K^e_{20 \times 20}] - [K^e_{20 \times 5}] [K^e_{5 \times 5}]^{-1} [K^e_{5 \times 20}] \\ \{\delta^n_{20 \times 1}\} &= \{\delta^e_{20 \times 1}\} \\ \{P^n_{20 \times 1}\} &= \{P^e_{20 \times 1}\} - \left([K^e_{20 \times 5}] [K^e_{5 \times 5}]^{-1} \{P^e_{5 \times 1}\} \right) \end{aligned}$$

3.4. Consistent Load Vector

The external applied nodal loads considered in the present finite element analysis are calculated by using a consistent load vector. The consistent load vector is obtained by equating the work done by the nodal loads on the nodal displacements to the work done by the external applied load on the assumed displacement function of the element. The double curved shell element shown in figure (1) is considered. If the element is subjected to several loading types such as, distributed normal pressure (q), concentrated load (Pi), uniform distributed moment (M), and concentrated moment (Mi) the load vector becomes:

$$\{P^e_{25 \times 1}\} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} [A]^{-1} [f] \begin{Bmatrix} q \\ m \end{Bmatrix} dx dy + \int_{y_1}^{y_2} \int_{x_1}^{x_2} [A]^{-1} [f]_{at \ node \ (i)} \begin{Bmatrix} P_i \\ M_i \end{Bmatrix} dx dy \tag{15}$$

For the element stiffness matrix after condensation, the load vector is taken as follows:

$$\{P^n_{20 \times 1}\} = \{P^e_{20 \times 1}\} - \left([K^e_{20 \times 5}] [K^e_{5 \times 5}]^{-1} \{P^e_{5 \times 1}\} \right) \tag{16}$$

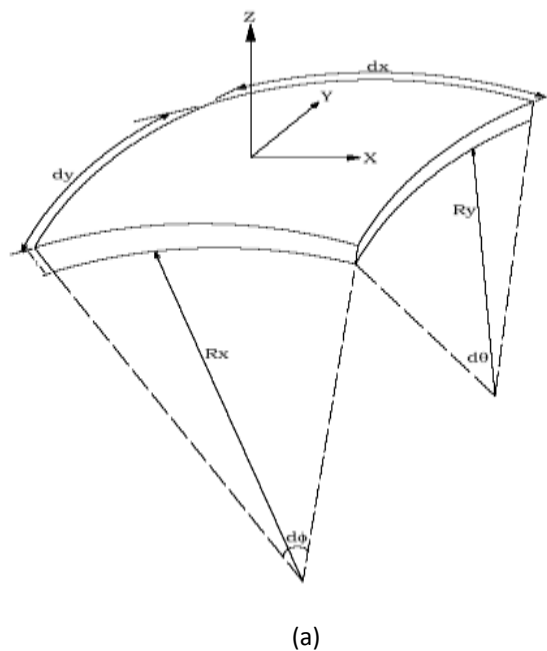


Figure (1) Geometry and in-plane, and out-of-plane displacements and stresses of doubly curved shell

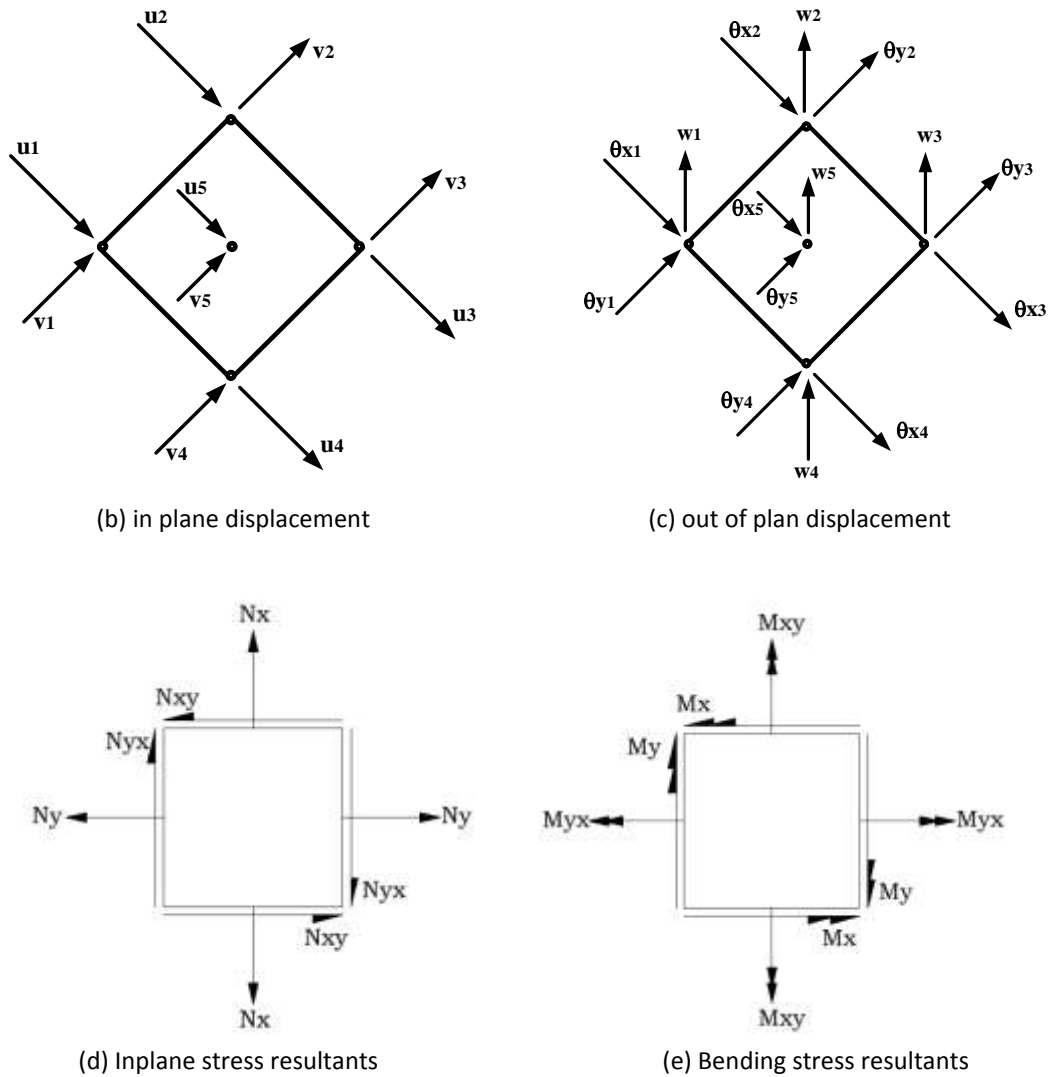


Figure (1) Continued

4. Clamped cylindrical shell under internal pressure

The present double curved shell element degenerates to a cylindrical shell element by putting the R_y very large ($R_y = 1010$), where (L_c) is equal to the cylindrical length. This problem represents a clamped-clamped cylindrical shell subjected to internal pressure as shown in Figure (2). As the problem is symmetric about its center line, only one half of the shell need be considered. Further in view of the axisymmetry in the problem, only a thin strip along the axis of the cylinder can be considered. In the present problem a strip generated with included angle of ($\theta=50$) is considered. One, two, three and four elements are used to idealize this thin strip. Solutions obtained for the maximum radial deflection for these idealizations are presented in Table (1) along with the analytical solution of Timoshenko and Woinowsky-Krieger[25] and finite element solution of Raju et. al [26]. This table demonstrates the rapid convergence and the accurate result obtained by the present element. The figure (3) shows the behavior of convergence of the present element and Raju's element with Timoshenko and

Woinowsky-Krieger[25]. It's appearing the present element is more accurate and has a rapid convergence than the Raju's element with respect to the exact solution of Timoshenko and Woinowsky-Krieger(25).

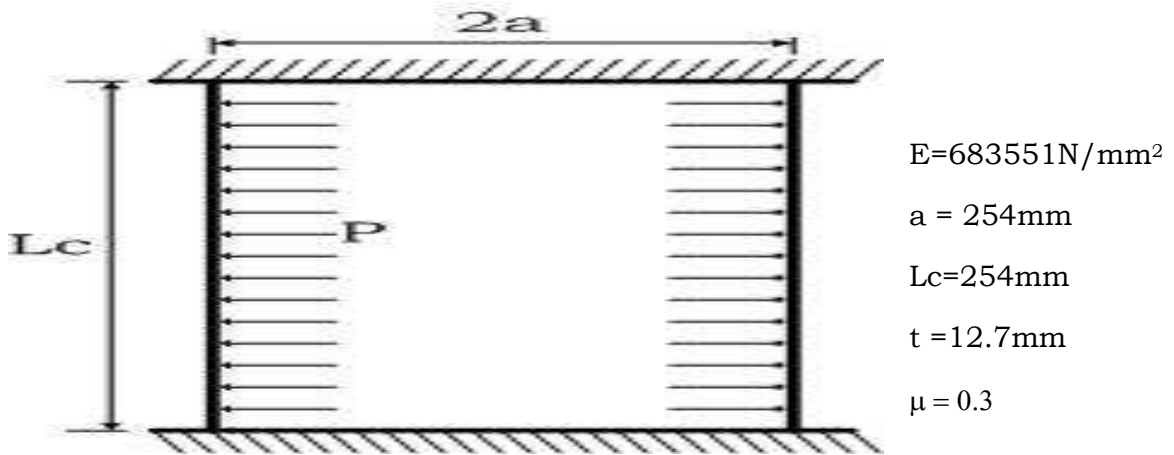


Figure (2) A Clamped-Clamped cylindrical shells under internal pressure.

Table (1) Maximum deflection of a clamped-clamped cylindrical shell under internal pressure x104.

Raju conical element			Present double curved shell element			(Timoshenko and Woinowsky-Krieger 1981)
Mesh size	D.O.F	Maximum radial deflection	Mesh size	D.O.F	Maximum radial deflection	
1x1	28	5.6312 mm	1x1	20	5.5422 mm	5.1486 mm
1x2	42	5.1333 mm	1x2	30	5.2334 mm	
1x3	56	5.1613 mm	1x3	40	5.1495 mm	
1x4	70	5.1410 mm	1x4	50	mm	

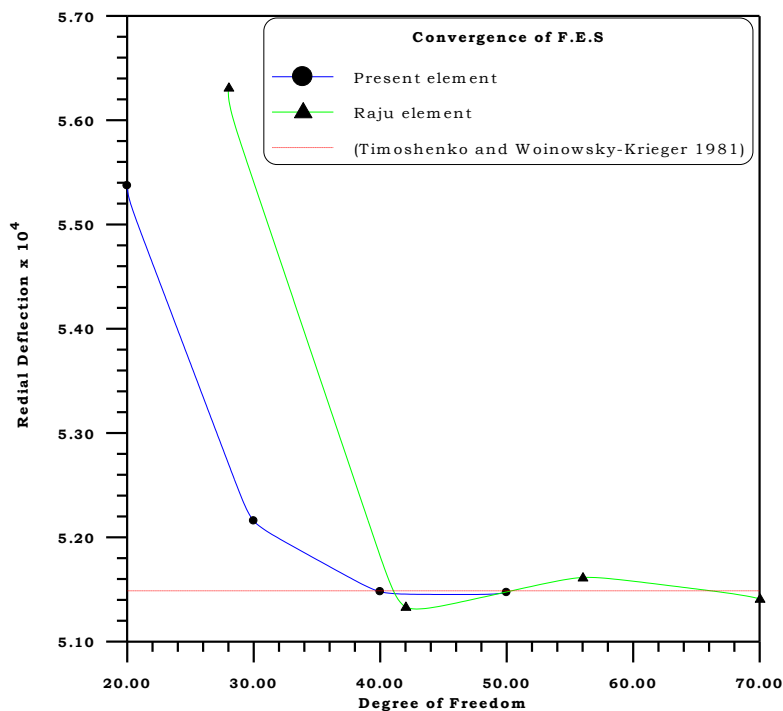


Figure (3) Relation between maximum radial deflection and d.o.f.

5. Pinched Cylinder Problem

A standard problem for testing non-axisymmetric cylindrical shell finite elements is the pinched cylinder shown in Figure (4). The most accurate solution to this problem appears to be that of Cantin and Clough[7], who used a procedure which satisfies the conditions for convergence to the correct result as the mesh size is reduced. Bogner et al. [6] and Cantin and Clough[7] did not continue the process of mesh refinement sufficiently, but were satisfied to obtain a value of displacement close to the in-extensional value of (2.7890 mm).

Timoshenko and Woinowsky-Kreiger[25] result, which is known to be too low, but not that of Cantin and Clough[7] did obtain one value of (2.7655 mm) by dividing an octant of the cylinder into three elements longitudinally and [49] circumferentially, with (1200) degrees of freedom. Also Ashwell and Sabir[27], and Raju et al. [1] analyzed this problem for a thick cylindrical shell with thickness (2.388mm) by using a cylindrical and conical element respectively. Ashwell and Sabir[27], and Sabir and Lock[9] analyzed the same problem but as a thin cylindrical shell with thickness (0.4mm).

This problem for both cases are analyzed by the present element, the comparison of the solution results of the present element and other elements are concluded in Table (2) for thick cylinder and Table (3) for thin one. Table (2) shows the results of five elements of thick pinch cylinder problem, the convergence of these elements are very good but the errors between results of these elements and exact solution of Timoshenko and Woinowsky-Krieger[25] are different from element to other. Table (3) shows the results of four elements of the thin pinched cylinder problem, the convergence of two elements by Ashwell and Sabir[27], and the present element are excellent but the other two elements, Cantin and Clough[7], and Sabir and Lock[9] show very poor convergence. The errors between results of these elements and exact solution of Flugge[24] are different from element to other.

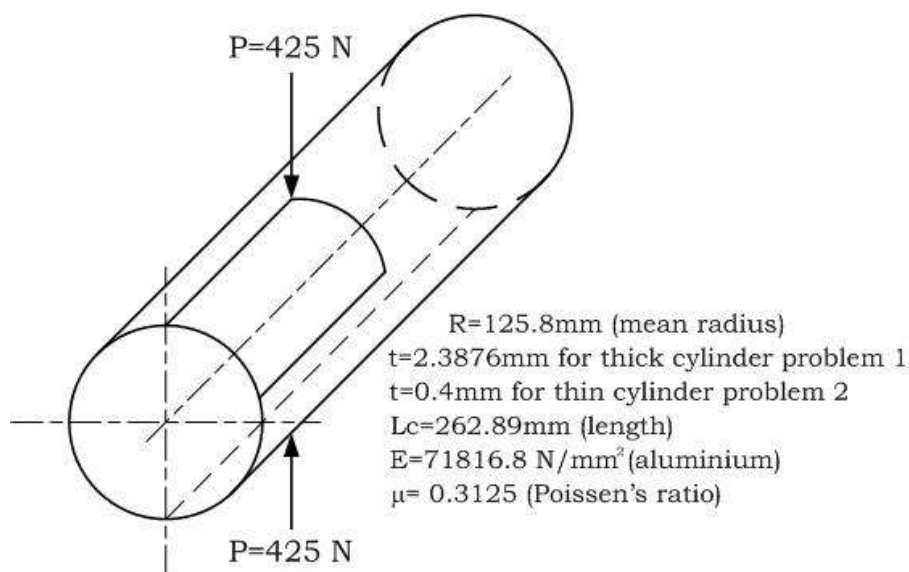


Figure (4) the pinched cylinder problem with one octant.

Table (2) Deflection in (mm) under one load for thick pinched cylinder problem ($t=2.3876\text{mm}$) prob. 1.

Mesh Size	Bogner et al (48 d.o.f)	Raju et al. (28 d.o.f)	Cantin and Clough (24 d.o.f)	Ashwell and Sabir (20 d.o.f)	Present Element (20 d.o.f)
1x1	-0.0635(48)*	-0.0356(28)*	-0.2964(24)*	-2.6416(20)*	-1.0431(20)*
1x2	-2.0371(72)*	-1.9469(42)*	-0.5789(36)*	-2.6824(30)*	-1.8928(30)*
1x3	-2.6060(96)*	-2.5491(56)*	-0.7544(48)*	-2.7345(40)*	-2.2764(40)*
1x4	-2.7610(120)*	-2.5392(70)*	-2.7915(60)*	-2.8092(50)*	-2.4597(50)*
2x2	-2.0523(108)*	-1.9741(63)*	-2.3647(54)*	-2.8016(45)*	-2.4889(45)*
2x3	-2.6314(144)*	-2.5715(84)*	-2.5070(72)*	-2.8219(60)*	-2.5789(60)*
2x4	-2.7890(180)*	-2.7605(105)*	-2.8270(90)*	-2.8372(75)*	-2.6241(75)*
3x3	-----	-2.5865(112)*	-2.6848(96)*	-2.8447(80)*	-2.6517(80)*
4x4	-----	-2.7799(175)*	-2.8600(150)*	-2.8677(125)*	-2.7546(125)*
6x6	-----	-2.8410(343)*	-2.8880(294)*	-2.8829(245)*	-2.7599(245)*
8x8	-----	-2.8603(567)*	-2.8931(486)*	-2.8880(405)*	-2.7654(405)*
10x10	-----	-2.8664(847)*	-2.8931(726)*	-2.8880(605)*	-2.7654(605)*
Exact Solution of (Timoshenko and Woinowsky-Krieger 1981) = -2.7655 mm					

*Value between brackets is total degrees of freedom

Table (3) Deflection in (mm) under one load for thin pinched cylinder problem ($t=0.4\text{mm}$) problem 2.

Mesh Size	Sabir and Lock (20 d.o.f)	Cantin and Clough (24 d.o.f)	Ashwell and Sabir (20 d.o.f)	Present Element (20 d.o.f)
1x1	-0.0003	-0.0003	-0.5844	-0.5749
1x4	-0.0160	-0.0188	-0.6104	-0.6017
1x8	-0.1755	-0.1778	-0.6111	-0.6087
2x1	-0.0003	-0.0003	-0.5842	-0.6107
2x4	-0.0163	-0.0178	-0.6119	-0.6129
2x8	-0.1762	-0.1775	-0.6132	-0.6148
3x1	-0.0003	-0.0003	-0.5847	-0.6157
3x4	-0.0165	-0.0173	-0.6132	-0.6171
3x8	-0.1767	-0.1775	-0.6142	-0.6180
8x8	-0.1795	-0.1800	-0.6175	-0.6191
Exact Solution of (Flugge 1973) = -0.6195 mm				

6. Cylindrical Tank with Constant Thickness

A cylindrical tank, Figure (5), with constant wall thickness is fixed at the base, and is completely filled with water of density 9.81kN/m^3 . The dimensions of the tank are as follows: height of tank ($L_c=7.925\text{m}$), radius of tank ($R=9.144\text{m}$), wall thickness ($t=356\text{mm}$), elastic modulus ($E=205\text{GPa}$) and Poisson's ratio ($\mu=0.25$) The present element is used to analyze this problem by taking a longitudinal strip of unit width and mesh size (1x9) elements and total degrees of freedom equal to (100).

Results obtained from this analysis, together with those given by Timoshenko and Woinowsky-Krieger(25), Thevendran[28], and Francis et al.[29] are summarized in Table (4 a, b, and c). The values given by Thevendran[28] were obtained by using the Runge-Kutta method with 100 equal steps, whilst those given by Timoshenko and Woinowsky-Krieger[25] were obtained by considering the simplifying assumption that the cylinder was treated as a long shell that is $\beta L_c > 5$ where $\beta^4 = 3(1 - \mu^2)/(R^2 L_c^2)$, and Francis[29] results were obtained by using the finite element analysis based on an

analogy with theory of beams on elastic foundation developed for closed circular cylindrical shells. In the present element the results of different stress resultants (M_s), (N_s), and (Q_s) are compared with the values of different approaches for analysis of cylindrical tanks and show excellent agreement with them.

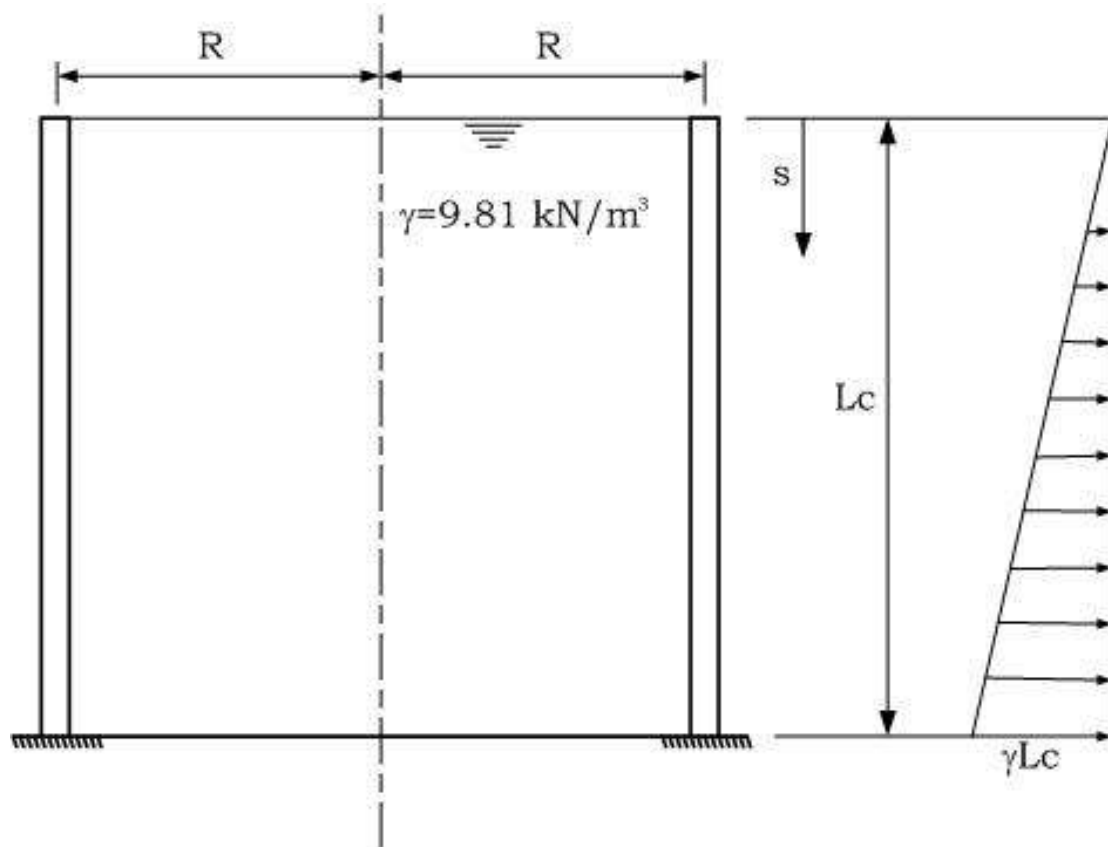


Figure (5) Cylindrical tank under hydrostatic loading.

Table (4a) Results of comparison of meridional normal force (N_s) (kN/m) for different approaches.

Ratio S/Lc	Timoshenko and Woinowsky-Krieger Solution	Thevendran Solution	Francis Solution	Present Element Solution
1.0	000.0	000.0	000.0	000.0
0.9	122.0	122.0	122.0	122.0
0.8	302.6	302.6	303.6	302.6
0.7	409.7	409.8	409.8	409.8
0.6	428.0	428.2	428.7	428.2
0.5	385.0	385.0	385.4	385.4
0.4	312.1	312.8	312.7	312.8
0.3	230.3	231.0	231.0	231.0
0.2	149.5	149.5	149.3	149.4
0.1	72.6	70.2	70.3	70.2
0.0	-0.9	-8.0	-7.7	-7.8

Table (4b) Results of comparison of meridional bending moments (Ms) (kN.m/m) for different approaches.

Ratio S/Lc	Timoshenko and Woinowsky-Krieger Solution	Thevendran Solution	Francis Solution	Present Element Solution
1.0	62.2	62.2	62.5	62.2
0.9	6.7	6.7	6.7	6.7
0.8	-13.6	-13.6	-13.7	-13.6
0.7	-15.1	-15.1	-15.1	-15.1
0.6	-10.0	-10.0	-10.0	-10.0
0.5	-4.7	-4.7	-4.7	-4.7
0.4	-1.3	-1.3	-1.3	-1.3
0.3	0.3	0.1	0.2	0.2
0.2	0.7	0.4	0.4	0.4
0.1	0.6	0.2	0.2	0.2
0.0	0.3	0.0	0.0	0.0

Table (4c) Results of comparison of meridional shearing force (Qs) (kN/m) for different approaches.

Ratio S/Lc	Timoshenko and Woinowsky-Krieger Solution	Thevendran Solution	Francis Solution	Present Element Solution
1.0	-98.8	-98.8	-98.5	-98.8
0.9	-45.0	-44.2	-44.2	-44.2
0.8	-11.3	-10.5	-10.5	-10.4
0.7	3.7	4.1	4.1	4.1
0.6	7.1	7.3	7.4	7.3
0.5	5.6	5.6	5.6	5.6
0.4	3.1	3.0	3.0	3.0
0.3	1.1	0.9	1.0	1.0
0.2	0.1	-0.1	-0.1	-0.1
0.1	-0.3	-0.4	-0.4	-0.4
0.0	-0.3	0.0	0.0	0.0

Table (5) Shows the error between present element and analytical solution of several problems.

Problem type	Error of present element
Clamped cylindrical shell under internal pressure	0.03%
Thick pinched Cylinder Problem	0.01%
Cylindrical Tank with Constant Thickness	0.0%

7. Conclusions

A doubly curved shell finite element which is suitable for the general bending analysis of shells has been developed. The element is simple and contains only the essential degrees of freedom. The element has the advantage over the other available double curved element. The improvement obtained is due to the fact that all the displacement fields of the present element satisfy the exact representation of rigid body modes of displacements. Also, the displacement fields due to straining of the element are based on independent strains and satisfy the exact compatibility equations of strain modes. The present element is used to analyze several types of cylindrical shell

problems. The numerical results of the present element are compared with the analytical, numerical, and experimental results of other researchers. The results of the present element showed good and rapid convergence of displacements and stresses with the use of few elements. The errors of output results are about less than 1% of mesh size (6x6) and less than 0.03% of mesh size (8x8) for static analysis of cylindrical problems.

Abbreviations

[A]	transform matrix
E	modulus of elasticity
R_x, R_y, R_{xy}	Principle radii of double curved shell
u, v, w	displacement field in X, Y, Z direction respectively
ur, vr, wr	rigid body mode displacement field in X, Y, Z direction respectively
us, vs, ws	strain mode displacement field in X, Y, Z direction respectively
γ	unit weight of water
ν	Poisson's ratio
θ_x, θ_y	rotation about x, and y axes respectively
$\varepsilon_x, \varepsilon_y$	the inplane direct strain in the direction x, and y respectively of shell

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