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Stability Coefficients Charts for Two Layered Earth Slopes Using Finite Element Method

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Abstract

Study the stability of slopes is of great importance for the geotechnical engineers. Stability analysis is required for engineering projects, such as natural stabilized slopes, embankments and cuts in road and excavation in soil. There are many methods used to analyze the slope stability problem. Limit equilibrium is most conventional method, which is used to analyze slope stability, which depends essentially on assuming a failure surface.

In this research, the finite element method is used to analyze slope stability problems. A finite element program is used to analyze the soil slope stability according to the theory of elasto-plastic failure of visco-plastic method. Mohr-Coulomb theory is used to represent the surface failure. The study concentrates on computing the factor of safety for stability at different values of cohesion (C) and angle of internal friction (φ) of the soil.

Comparison between the computer results with the traditional slip circle solution gave good agreements. The study also considered soil slopes in two layers with different thickness and shear strength. Wide range of angle of slope is used in this research. Finally, charts for coefficient of slope stability are introduced with different ratio of cohesion for two layered soil different in thickness.

الخلاصـــــــة

يعدّ إستقرار المنحدرات الأرضية من الموضوعات التي لها أهمية كبيرة بالنسبة لمهندسي التربة. مشاريع كثيرة تتطلب تحليل إستقرارِ المنحدرات، مثل المنحدرات الأرضية الطبيعية المثبتة، والاملاءات والقطوع التي تستخدم لإنشاء الطّريقِ وكذلك في أعمال الحفر. هناك العديد من الطّرقِ التي تستخدم لتُحليّلُ إستقرار المنحدرات الأرضية. طريقة التوازن الحدي هي الطّريقةُ العامة المستخدمة لُتحليّلُ المنحدرات والتي تَعتمدُ بشكل أساسي على سطح الفشلِ المُفتَرِض.

في هذا البحثِ تم استخدام طريقةِ العناصرِ المحددةِ لتحليل مشكلة إستقرار المنحدرات الأرضية. برنامج العناصر المحددة الذي استخدم في تحليل إستقرار منحدرَ التّربةِ قَدْ اعتمد على طريقة الفشل المرن-اللدن، كذلك تم استخدام نظرية مور-كولومب لتُمثّيلَ سطح الفشل. رَكّزتْ الدّراسةُ على حِسابِ عاملِ الأمانِ لقيم مختلفة من خصائص التربة(التماسك والاحتكاك).

مقارنة النتائج التي تم الحصول عليها من البرنامج مع الحلول التقليدية أعطى توافق جيد. اهتمت الدراسة أيضاً بالترب ذات الطبقتين المختلفتين بالسمك. تم استخدام مدى واسع من زوايا الانحدارات الأرضية. أخيراً، تم تقديم مخططات لمعاملات الاستقرارية لنسب مختلفة من المقاومة ولترب ذات طبقتين مختلفتين بالسمك.

1. Introduction

There are many methods used to analyze the soil slope stability. The stability of slopes is assessed by determining the safety factor that is depending on the properties of soil, which represent the strength to failure. The resistance to failure depends on cohesion (C) and angle of internal friction (φ). When the ground surface is sloping, forces are generated. The important forces induced in the slopes are the force of gravity and the force of seepage water, which induce shearing stresses in the soil. In practice, limiting equilibrium methods are used in the analysis of slope stability. It is considered that failure occurs at any point along the failure surface. The failure surface is assumed according to the type of slope.

There are several types of surface failure. Circular, noncircular rotational slip, transition and compound slip may be the potential surface failure. Suggestion of a surface failure depends mainly on the homogeneity and the strength of layered soil. The most conventional methods that are used to analyze slope stability problems are the φ-circle method and the Slices methods. The principle of slices method is used widely by many researchers. The object is to simplify the solution that is needed by much iteration and to make the method valid for any surface failure shape. The original concept of this method was developed by Fellenius, and then by ***(Bishop, 1955)***. ***Bishop and Morgenstern (1960)*** published dimensionless stability coefficients for homogeneous slopes. ***Morgenstern and Price (1965)*** developed a general analysis in which all boundaries and equilibrium conditions are satisfied and in which the failure surface may be of any shape. ***Spencer (1967)*** proposed a method of analysis in which a numerical solution is used and showed that the accuracy of Bishop's simplified method is satisfied. ***Bell (1968)*** proposed a method in which the soil mass is considered as a free body as in the case of φ-circle method. ***Michalowski (2002)*** used another method to indicate the safety factor; he presented stability charts for uniform slopes based on the kinematics approach of limit analysis. In all the previous methods of analysis a surface failure should be proposed. This means that the analysis depends essentially on the proposed slip surface. Real analysis is performed when the soil is used as a nonlinear material and the differential equations that govern the problem are used. Solution for the differential equations can be conducted by the finite element method. Reluctance to use finite element method for slope stability analysis in practice has been partly due to concerns that it is complex and computationally time-consuming ***(Lane and Griffiths, 1997)***. With the developing of computers in both hardware and software fields the finite element methods become commonly used. The essential program that is used in this research to analyze slope stability problems was taken from ***(Smith and Griffiths, 1998)***. ***Griffiths and Lane (1999)*** developed further the program to take into account the water seepage effect. The program has been used to analyze several slope stability problems including the influence of layering and free surface on slope and dam stability. ***Chok et. al. (2000)*** studied the effect of vegetation on stability of slopes. It was concluded that the vegetation reduces the pore water pressure and increases the soil shear strength. The effects of soil suction and root reinforcement has been quantified as an increase in apparent soil cohesion. The effect of root reinforcement is considered by using an apparent root cohesion (Cr = 5 kPa). The depth of root zone (hr) is considered to be

1 meter. The program that is used in this study is the same program that is used to study the effect of vegetation. In this research the effects of internal friction and the cohesion were studied.

In practice the strength of soil is not constant with depth and the soil is not of the same properties with depth, it may be in two or more layers. There are a few researches, which deals with slopes of two layers and the cases that have been studied are limited. For the case of two layers with different strength, factor of safety is obtained by computing the average value of strength, and then using the classical charts to find the safety factor. In this research charts are introduced to compute the factor of safety for two layered slopes depending on easy equations. The charts introduced here are based on the results that obtained from the program of finite element without any averaging to the strength parameter. This study will explain the difference between the results of safety factor when the strength of the two layers is averaged and when the strength is used as a real case. The introduced charts give accurate results with confidence compared to the other charts.

2. Model of Slope Stability Problem

2-1 Finite Element Modeling

The finite element model in the present study assumes two dimensional plane strain conditions. The program that is used in this study uses nonlinear element of eight-node quadrilateral elements. The suitable failure criterion that represents the soil that possessing frictional and cohesion components of shear strength is Mohr-Coulomb criterion. When the research deals with undrained soil Von Mices theory is more convenient. Failure of the slopes can be defined in different ways ***(Abramson et. al., 1996)***. Factor of safety of slopes may be computed by using the non-convergence solution, coupled with a sudden increase in nodal displacements as an indication of failure condition ***(Griffiths and Lane, 1999)***. Another approach which is depended on is the failure of the visco-plastic algorithm converging within an iteration limit usually “250 iterations”, with nodal displacement criterion on successive iteration ***(Lane and Griffiths, 1997)***. For a successive iteration a tolerance of (0.0001) in this program enables the iterations to be stopped when successive solution are close enough but since iteration is a loop which could carry on “for ever”, a maximum number of iterations is specified to 250 iteration. Selection the value of maximum number of iterations of 250 iterations is not a rule also the tolerance. Comparing the computer result with traditional solution assesses them.

2-2 Factor of Safety

The factor of safety (FoS) of the slope is to be assessed, and this quantity is defined as the proportion by which (tan φ) and (C) must be reduced in order to cause failure. The factored soil strength parameters that go into the elasto-plastic analysis are obtained from:

$$\phi_f = \tan^{-1}\left(\tan\phi / FoS\right) \quad \text{.......... (1)}$$

$$C_f = C / FoS \quad \text{.......... (2)}$$

where:

ϕ_f: factored friction angle

ϕ : friction angle

C: cohesion

C_f: factored cohesion

Several (usually increasing) values of the factor of safety are attempted until the algorithm fails to converge. The actual factor of safety of the slope is the value to cause failure.

3. Governing Equations

3-1 Visco-Plasticity

In this method ***Zienkiewicz and Cormeau (1974)*** assumed that the material is allowed to sustain stresses outside the failure criterion for finite “periods”. The visco-plastic strain rate is given by the following equation:

$$\varepsilon^{\bullet VP} = F \frac{\partial Q}{\partial \sigma} \quad \text{.......... (3)}$$

The derivative of the plastic potential function Q with respect to stresses are expressed through the chain rule; as following:

$$\frac{\partial Q}{\partial \sigma} = \frac{\partial Q}{\partial \sigma_m}\frac{\partial \sigma_m}{\partial \sigma} + \frac{\partial Q}{\partial J_2}\frac{\partial J_2}{\partial \sigma} + \frac{\partial Q}{\partial J_3}\frac{\partial J_3}{\partial \sigma} \quad \text{.......... (4)}$$

where:

$$J_2 = \frac{1}{2} t^2 , \qquad J_3 = s_x \, s_y \, s_z - s_z \;\; \tau^2{}_{xy}$$

The above equation can be solved numerically by an expression of the form:

$$\varepsilon^{\bullet VP} = F\,(\,DQ_1\,M^1 + DQ_2\,M^2 + DQ_3\,M^3\,)\,\sigma \quad \text{.......... (5)}$$

where:

M^1, M^2, M^3 are given in Appendix (1).

Multiplication of the visco-plastic strain rate by a pseudo-time step gives an increment of visco-plastic strain which accumulates from one "time step" or iteration to the next; thus:

$$(\Delta\varepsilon^{VP})^{i} = (\Delta\varepsilon^{VP})^{i-1} + \Delta t(\dot{\varepsilon}^{VP})^{i} \quad \text{..........} (6)$$

where, the time step for Mohr-Coulomb materials as derived by ***(Cormeau, 1975)*** is:

$$\Delta t = \frac{4(1+\upsilon)(1-2\upsilon)}{E(1-2\upsilon+\sin^2\phi)} \quad \text{..........} (7)$$

3-2 Mohr-Coulomb Failure Criterion

Algebraically, the surface of failure is expressed in terms of failure function F. This function, which has units of stress, depends on the material strength and invariant combinations of the stress components. Failure function can be defined as following:

$$F = \sigma_m \sin\phi + \bar{\sigma}\left(\frac{\cos\theta}{\sqrt{3}} - \frac{\sin\theta\,\sin\phi}{3}\right) - c\cos\phi \quad \text{..........} (8)$$

where:

$$\sigma_m = \frac{s}{\sqrt{3}}, \quad \bar{\sigma} = t\sqrt{3/2}, \quad \theta = \frac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{6}\,J_3}{t^3}\right),$$

$$t = \frac{1}{\sqrt{3}}\left[(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6\tau^2_{xy}\right]^{\frac{1}{2}}$$

4. Verification

4-1 Problem Definition

The study assumes two-dimensional plane conditions. An elasto-plastic model with Mohr-Coulomb failure criterion is assumed. The essential program used in this study was developed by ***Smith and Griffiths (1998)*** and it uses eight nodded quadrilateral elements. **Figure (1)** shows the mesh for a typical slope stability analysis. The shape of the mesh is like a trapezium with the restriction that the top and bottom boundaries are parallel to the x-axis. The mesh is considered as 25 elements density. Elements of finite element grid are different in dimension from one to another. Properties of the soil are listed in **Table (1)**. The output of the program gives the factor of safety, the maximum displacement at convergence and the number of iterations to achieve convergence.

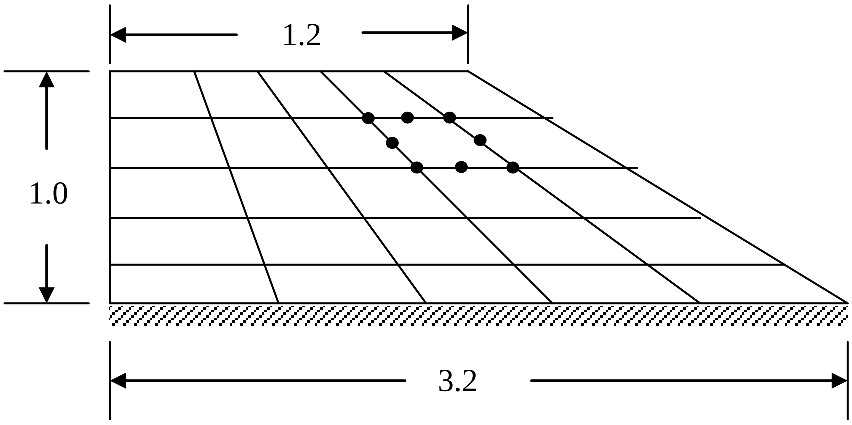


Figure (1) Typical soil slope mesh After Smith and Griffiths 1998

Table (1) Properties of studied soil

φ	C	ψ	γ	E	ν
40	1 kPa	0	20 kN/m3	100MPa	0.3

4-2 Verifying the Results

First, the program was checked by solving the same old examples that were solved by other methods and then the results are compared. The program of finite element is compared with the results of ***Bishop and Morgenstern (1960)*** for the same problem. **Figure (2)** show that FoS equal to 2.5 at 250 iteration which is the same result compared with traditional result.

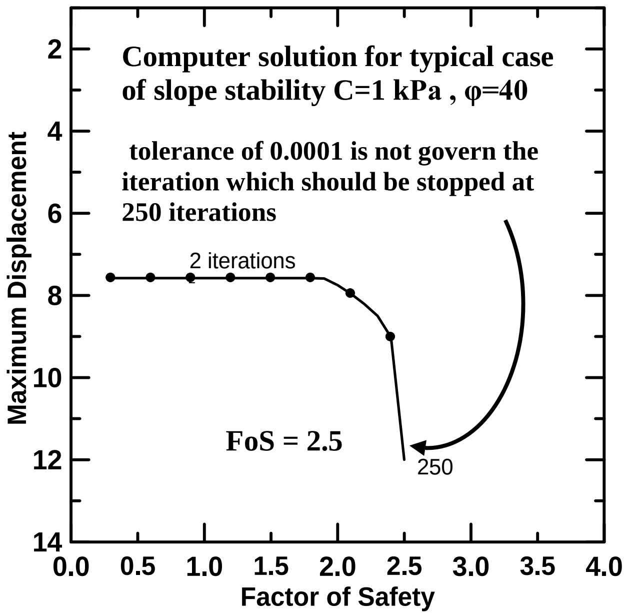


Figure (2) Safety factor versus maximum displacement for φ = 40 and C= 1 kPa

5. Effect of Cohesion and Angle of Internal Friction on the Factor of Safety

The problem is solved for different values of cohesion (C) and angle of internal friction (φ) to compute the effect of these parameters on the safety of the slope stability. Eighty factors of safety ranging from 0.5 to 10 are attempted. The results of the factor of safety, the maximum displacement at convergence and the number of iterations to achieve, are computed for each case study. Four cases of parameter (C) are taken (1, 5, 10, 15 kPa). For each case there are several cases of parameter φ ranging between (5-50). These values of cohesion (C) and angle of internal friction (φ) are selected as a specimen of cases that should be studied.

The aim is to show how the effect appears with increase of cohesion (C) and angle of internal friction (φ). The results give a good agreement with the ***(Bishop and Morgenstern, 1960)*** solution. **Figure (3)** shows the relation between factors of safety FoS with internal angle of friction (φ) for each case of c value. As obvious from the curves, the safety factor increases with increasing of φ for each case of cohesion (C). It can be noted from curves for φ = 40, C = 1 kPa that the FoS is equal to 2.5 while it is equal to 7 for C = 10. In general the increase in c and φ increases the safety factor. This fact is assessed by Coulomb relation:

$$\tau_f = c + \sigma \tan \phi \quad \text{.. (9)}$$

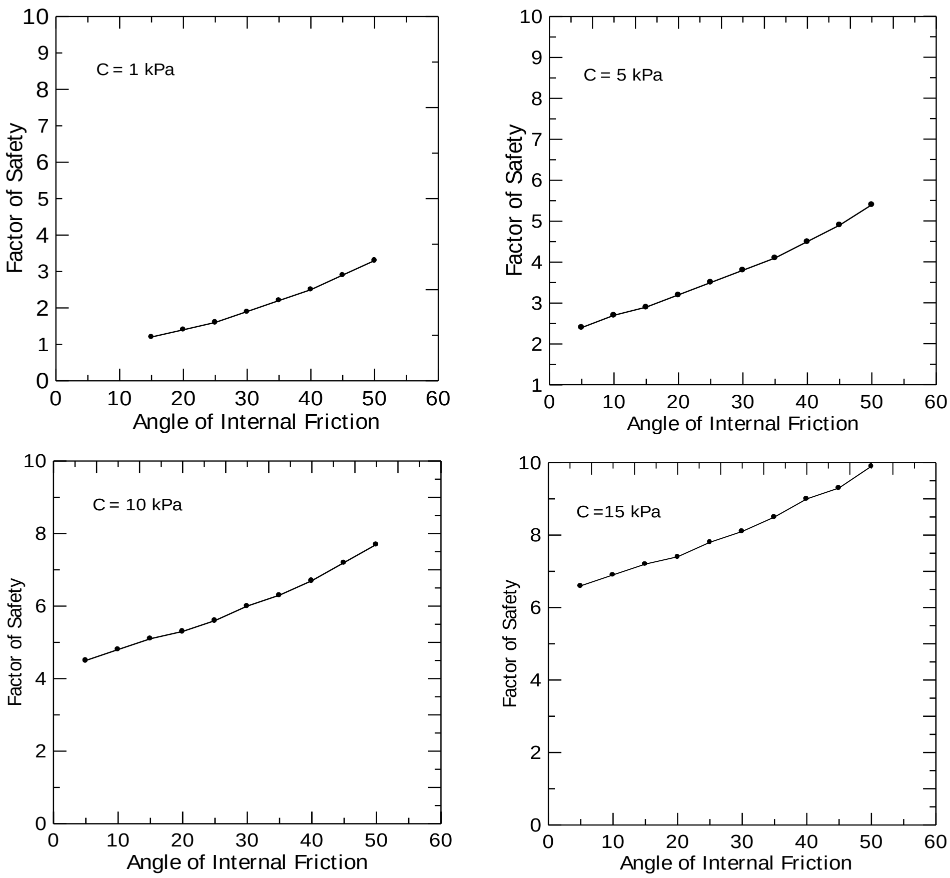


Figure (3) Relationship between angle of internal friction and stability factor of safety at C = 1,5,10 and 15 kPa

6. Effect of the Angle of Slope

Figure (4) shows a sketch for the problem that will be studied. The height of the soil slope is considered to be (4 m) and the angle of slope is equal to b (2, 3, 4 and 5) where:

$$\beta = \tan^{-1}\left(\frac{H}{x_2 - x_1}\right), \quad b = \cot(\beta)$$

Two values of cohesion C (2 and 4 kPa) are used for the case of C - φ soil. The program is used to compute the factor of safety for each value of c where the values of φ are different. The magnitudes of the angle of internal friction that used here are (40, 37.5, 35, 32.5, 30, 27.5, 25, 22.5, 20, 17.5, 15, 12.5, 10). Curves are drawn between the factor of safety and cot β for different value of φ for each $c/\gamma H$ value of (0.05, 0.025). **Figure (5)** shows the results of safety factor compared to the results obtained from the classical method of ***(Bishop and Morgenstern)***. The results give a good agreement.

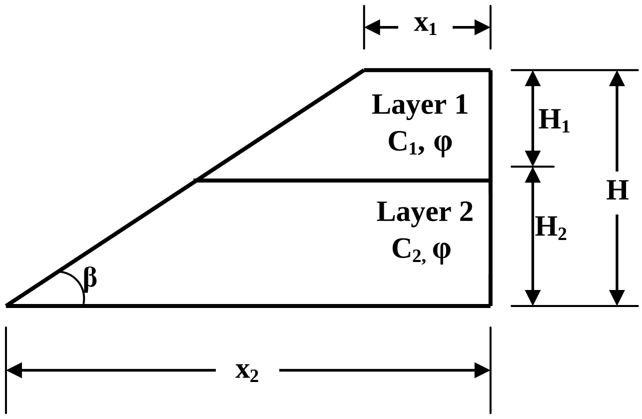


Figure (4) Sketch for the case of two layered soil slope stability

7. Two Layered Soil

Figure (4) states the problem of two layered soils, the thickness of the first layer is called H_1 and the thickness of the second layer is called H_2. Each layer has shear strength parameter C different from other layer. The program was developed to take into account the effect of the difference in the shear strength parameter C of the two layers on the factor of safety. φ parameter is taken as a constant while the C parameter is taken as a variable. Shear strength parameter C is included in the visco-plastic algorithm as an array of property for each element in the mesh where each element may has different value of C parameter. The program uses different ratios of C parameters (C_1/C_2) such as (10, 5, 2, 0.5, 0.2 and 0.1) with angle of internal friction equal to 0 and 10. The program gives factor of safety of two layered soil slope stability that has different value of shear strength at different ratio of shear strength.

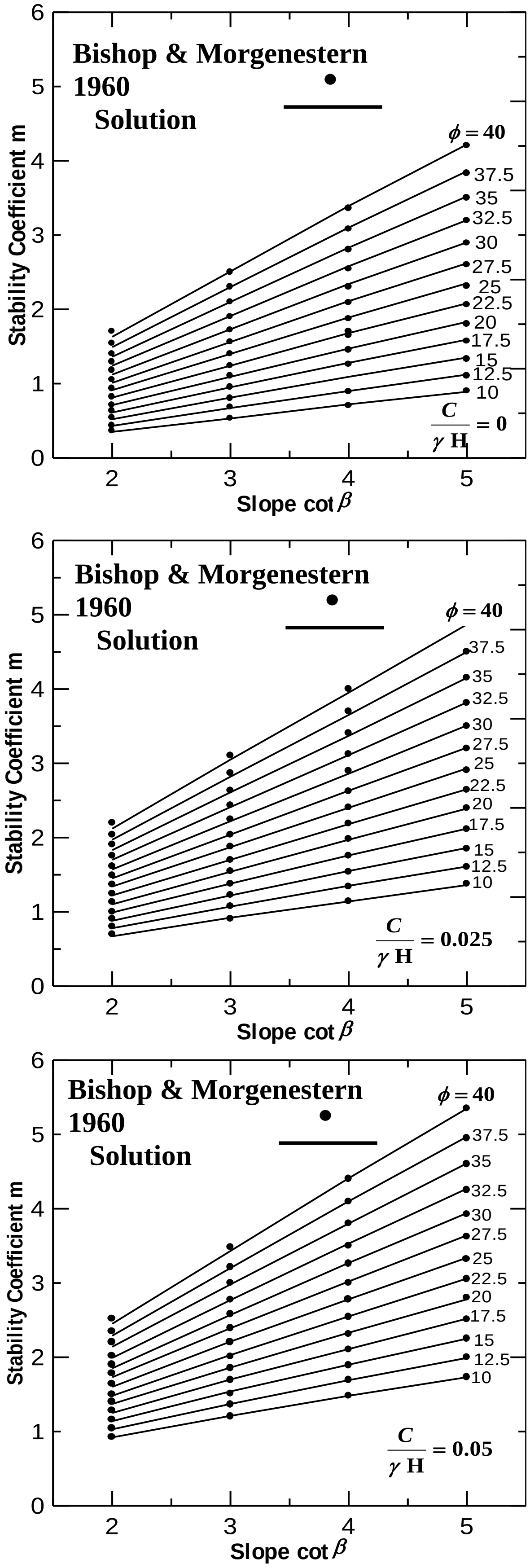


Figure (5) Relationship between the stability coefficient for earth slopes and slope angle

Figure (6) show the relationship between FoS and thickness ratio H_1/H at C_1/C_2 = 2 where C_1 and C_2 have different values. In general it can be seen that FoS increases with increasing the thickness of top layer which has greater shear strength than bottom layer. The type of increasing of FoS that can be noted from curves is linear, but it is varying with increasing the thickness ratio and it can be divided into three parts: first part is between H_1/H (0-0.2), Second between (0.2-0.8), and third between (0.8-1). The third part gives high increase while the first part gives lower increase.

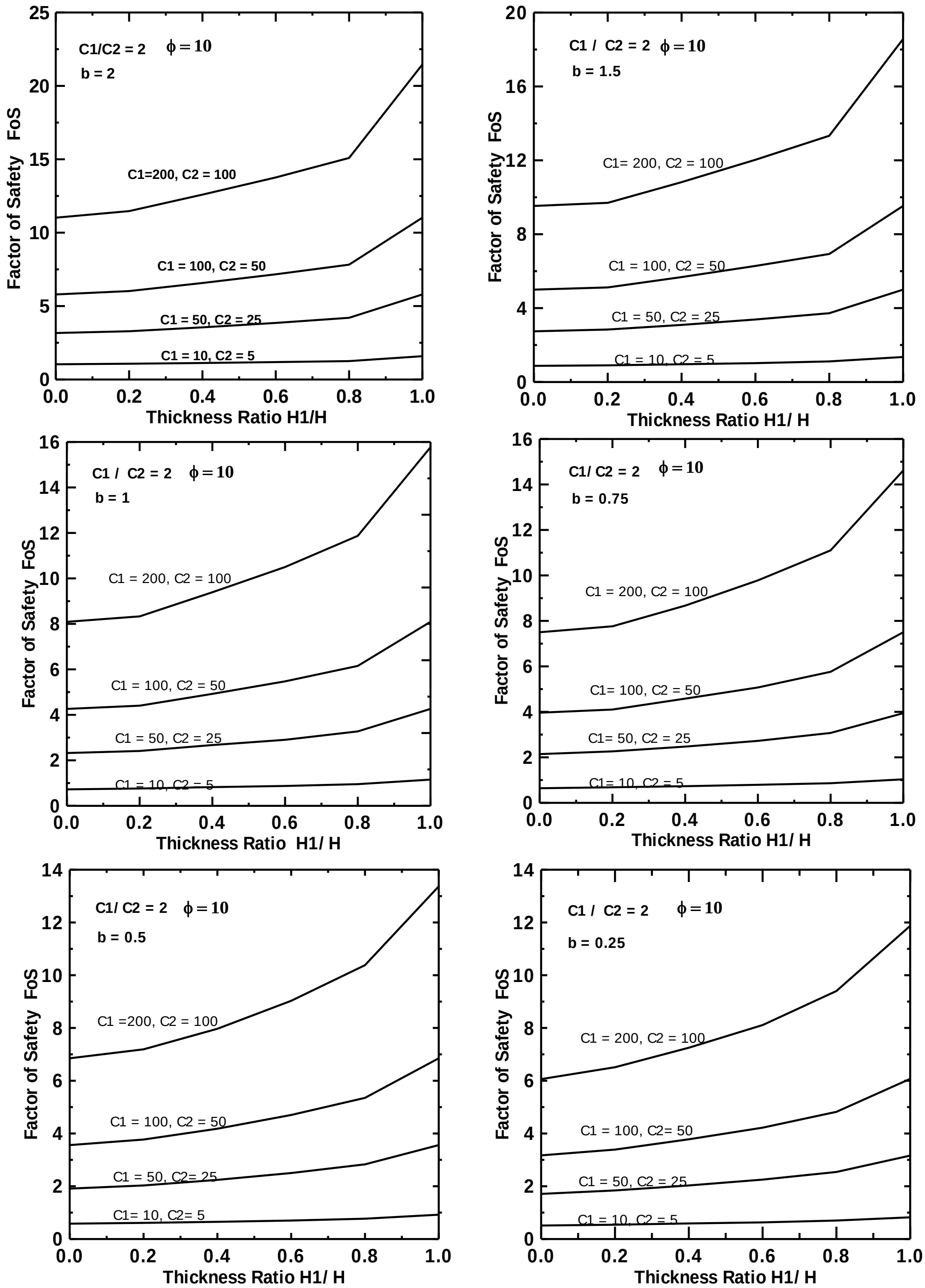


Figure (6) Relationship between factor of safety and thickness ratio at shear strength ratio C_1/C_2 =2 for different angles of slope

Figure (7) show the relationship between FoS and thickness ratio at $C_1/C_2 = 0.1$ where C_1 and C_2 have different value. It is obvious that FoS decreases with increase the thickness of top layer which has lower strength. Each curve can be approximately divided into two parts the first part at thickness ratio between (0-0.4) gives high decrease in safety factor while the second part give a low decrease in FoS. It can be seen from the view of sixth graph of **Fig.(7)** where $C_1/C_2=0.1$, b=0.25 the decreasing rate of FoS at first part equal to 20 while it is equal to 3.3 at second part where:

$$\textbf{Decreasing Rate of FoS at First Part} = \frac{FoS_{\frac{H_1}{H}=0.4} - FoS_{\frac{H_1}{H}=0}}{0.4-0}$$

$$\textbf{Decreasing Rate of FoS at Second Part} = \frac{FoS_{\frac{H_1}{H}=1} - FoS_{\frac{H_1}{H}=0.4}}{1-0.4}$$

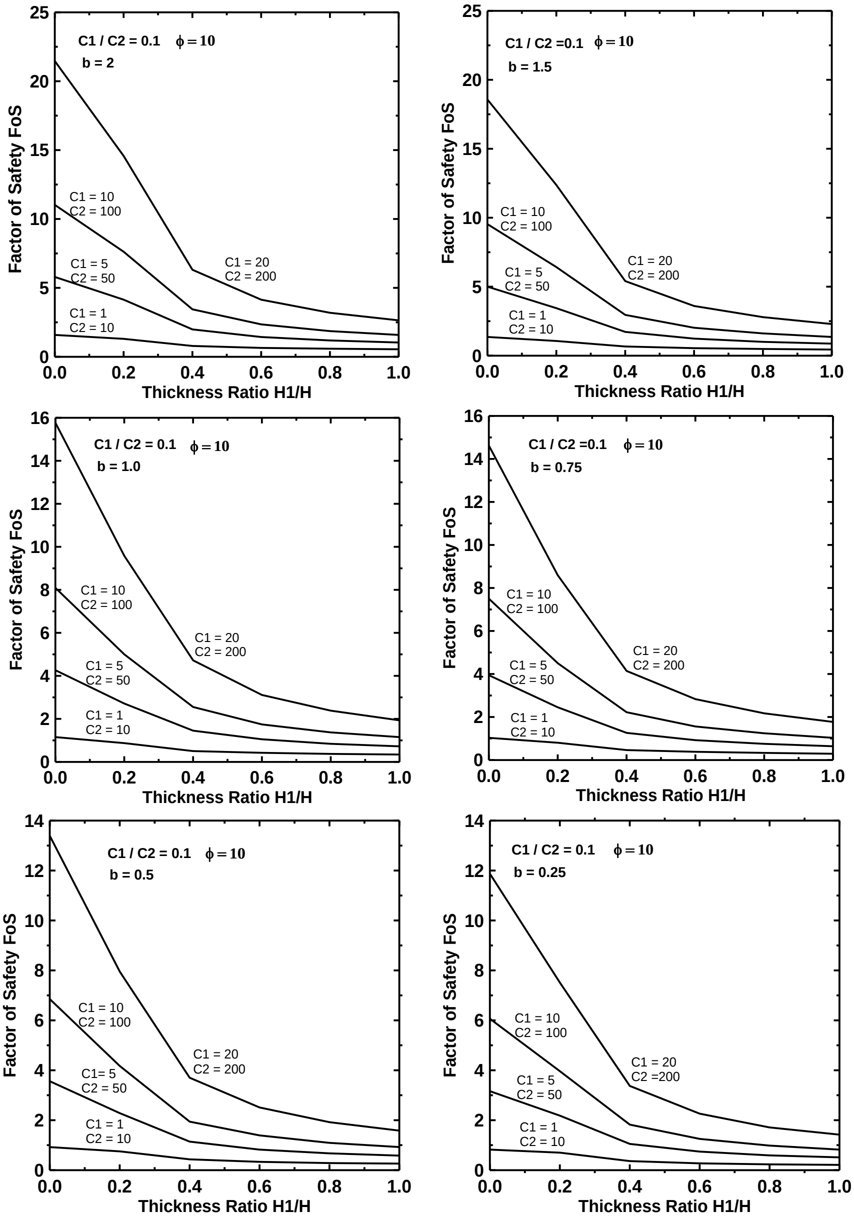


Figure (7) Relationship between factor of safety and thickness ratio at shear strength ratio C_1/C_2 =0.1 for different angles of slope

8. Charts of Stability Number

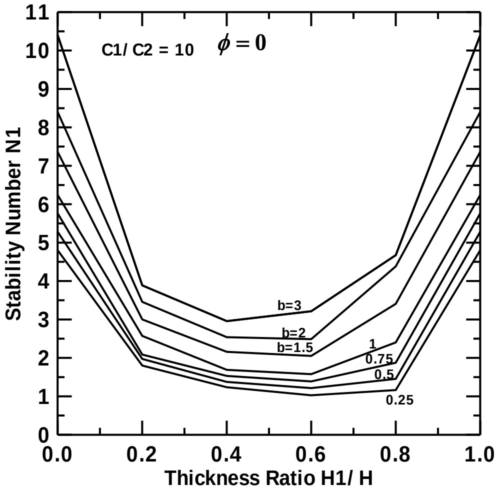
The study considers some cases of shear strength ratio such as C1/C2 = 0.1, 0.2, 0.5, 2, and 10. **Figure (6)** and **(7)** represent the relations of FoS and thickness ratio at C_1/C_2 =0.1 and 2. Results obtained from the program show that the FoS for each ratio of shear strength (C_1/C_2) is not constant and giving different values when the magnitude of C_1 and C_2 are varied. Representing all these result by curves are not convenient therefore; stability coefficients should be obtained and used with a simple equation to compute FoS. FoS is computed for each ratio with different magnitude of C_1 and C_2 such as at ratio C_1/C_2 = 2, the magnitude C_1=200, C_2=100 and C_1=100, C_2=50 and C_1=50, C_2=25. Curves are drawn between FoS and the factor *f* where:

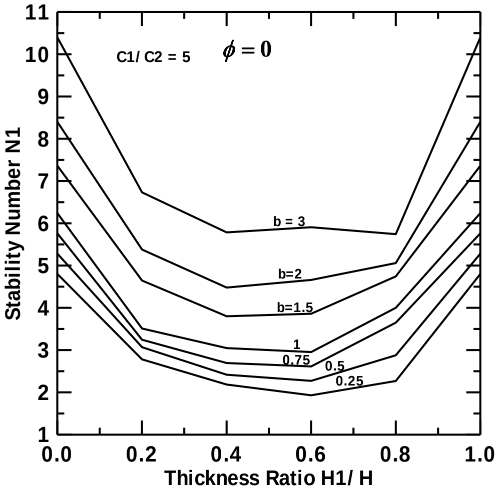
$$f = \frac{C_1 H_1 + C_2 H_2}{\gamma H^2} \quad \ldots\ldots (10)$$

The relation between FoS and *f* is linear. Curve fitting is used for each case of shear strength ratio to obtain the stability coefficient N_1 and N_2 of the following linear equation:

$$FoS = f\,N_1 + N_2 \quad \ldots\ldots (11)$$

It was found that N_2 factor for φ = 0 is equal to zero while N_1 factor has a value more than zero for φ = 0 or 10. **Figure (8)** shows the relationship between stability coefficient N_1 with thickness ratio for soil slope stability with two layers have different ratio of shear strength (C_1/C_2) and for φ=0. **Figures (9)** and **(10)** show the relationship between the coefficient of stability N_1 and N_2 with thickness ratio for the case of φ=10. These coefficients should be used in the above equation to find the factor of safety.





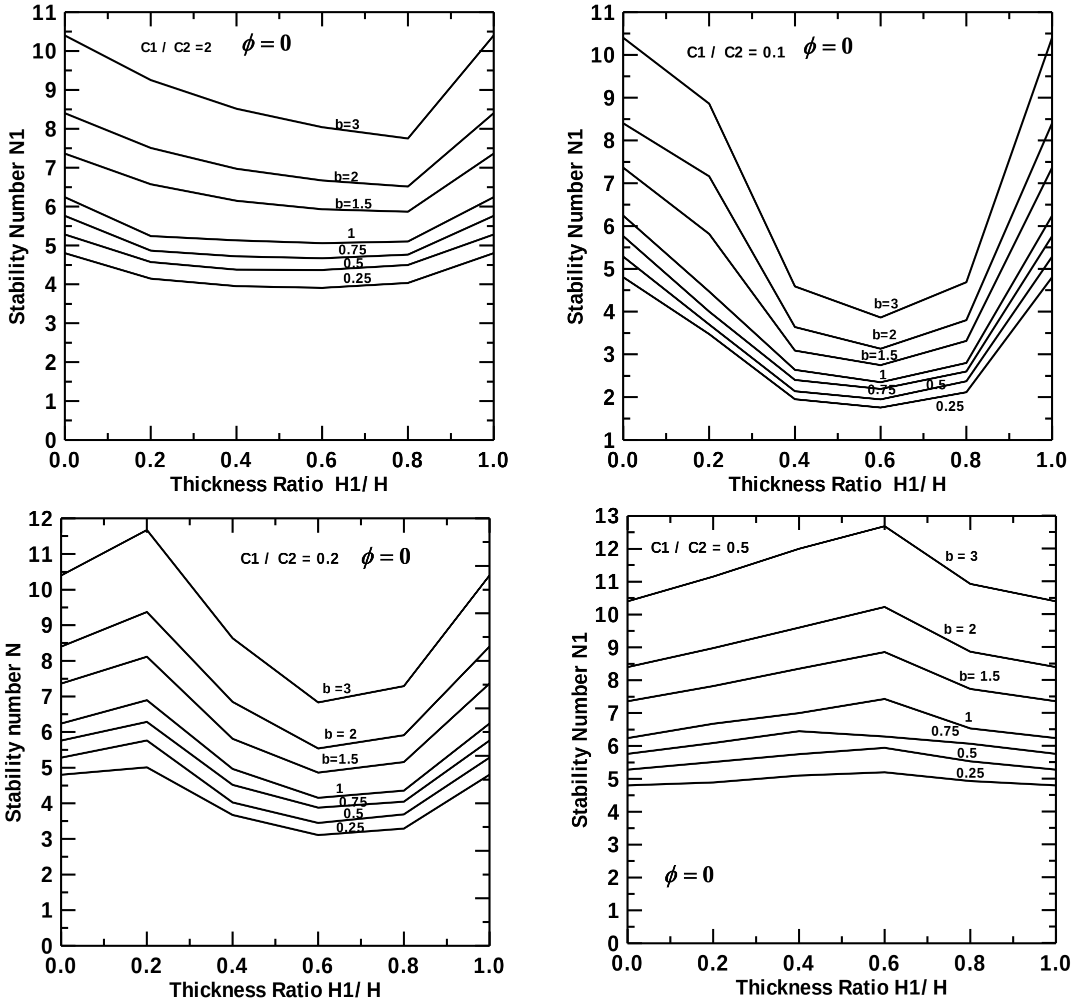
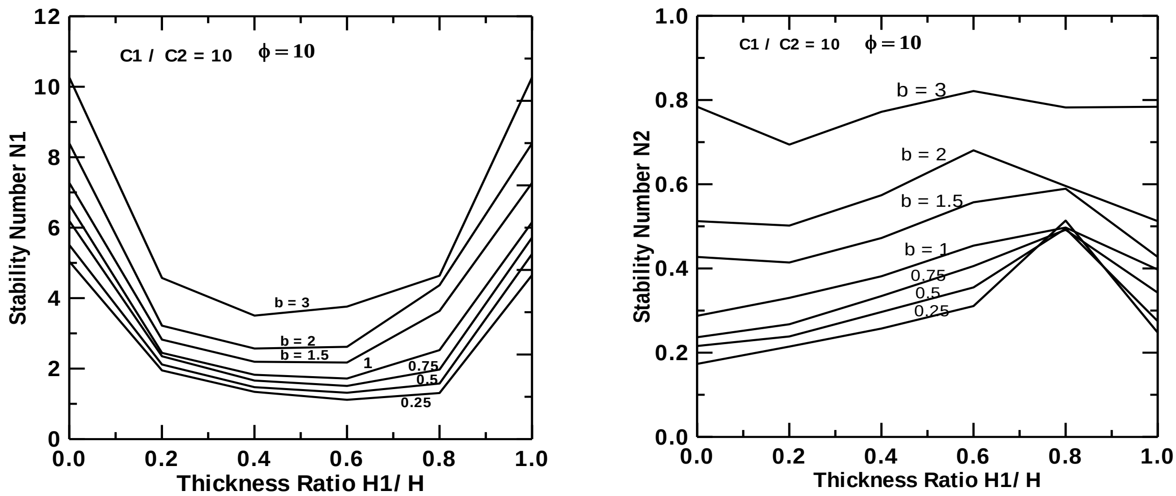


Figure (8) Relation between the stability number and the thickness ratio for different thickness of shear strength



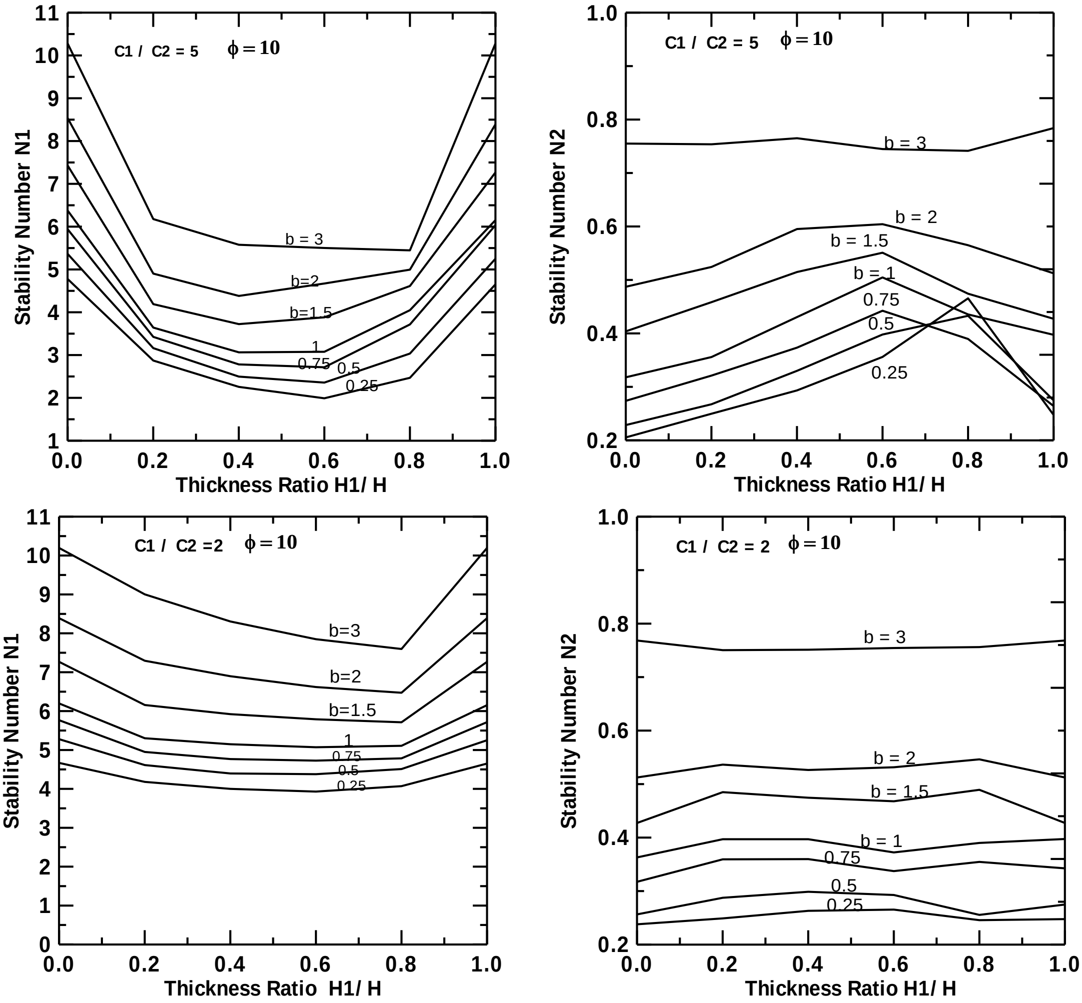
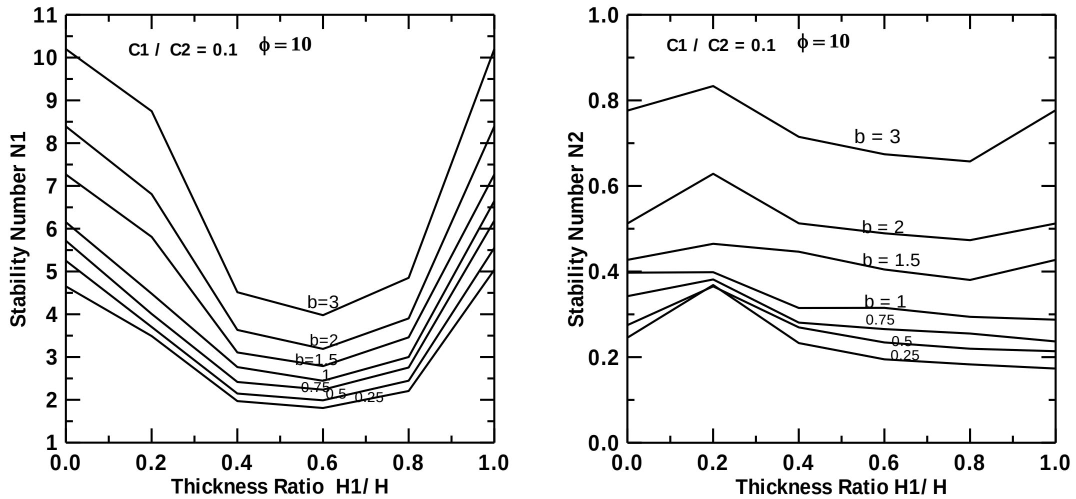


Figure (9) Relation between the stability number (N1, N2) and the thickness ratio for different ratio of shear strength where φ=10



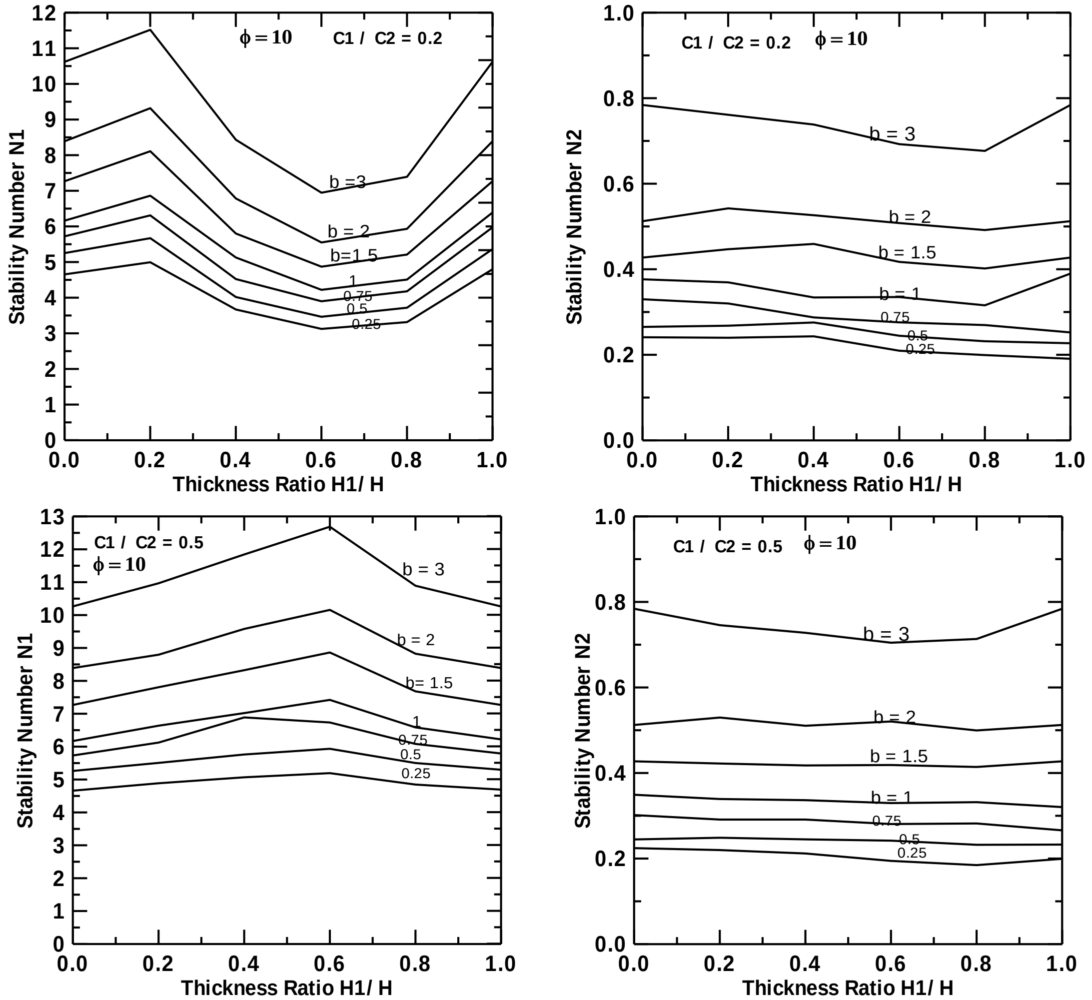


Figure (10) Relation between the stability number (N1, N2) and the thickness ratio for different ratio of shear strength where φ=10

9. Conclusion

1. In this study, the finite element method is used to analyze the slope stability problem. The study is concentrated on computing the safety factor for different value of C and φ that gives the strength to the slopes. As it is obvious, any increase in C or φ or both C and φ will give a good stability in slopes. The model is more close to the real case and can be used successfully in practice. The thought of stabilizing soil slopes depends on increasing the strength of soil C in spite of the type of improvement method. Results obtained from this study can help the designer to estimate the necessary work for stabilization.

2. A set of charts were produced to assess the stability of two layered soil slopes. The charts are used to find the safety factor for the stability according to the stability numbers and suggested equation. Different ratio of cohesion was used with constant internal friction. The charts can be used for φ = 0 and 10 or between these two values by interpolation. The stability numbers of slopes were obtained from the calculation based on the visco-plastic analysis using the Mohr Coloumb criterion.

3. Was concluded that the factor of safety for stability of two layered soil based on the average value of shear strength C is different from the factor of safety obtained by the program of finite element without averaging shear strength C. This difference should be taken in the computing the safety factor for soil slope stability.

4. The existence of soft layer at thickness ratio 0.8-1 has active effect on the safety factor compared to that at thickness ratio 0-0.8.

5. Factor of safety for two layered slope stability is decreases obviously when the top layer is soft and at thickness ratio 0-0.4. It is necessary to give more interest to the first layer which has lower value of shear strength.

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List of Symbols

β:	Angle of Soil Slope
b:	Cot β
C:	Cohesion
C_f:	Factored cohesion
E:	Modulus of elasticity
F:	Value of failure function (unit of stress)
FoS:	Factor of safety
Q:	Plastic potential function
s:	Mean stress
t:	Deviator stress
Δt:	Pseudo-time step
θ:	Lode angle
v:	Poisson ratio
$\sigma_1, \sigma_2, \sigma_3$:	Principal stresses
σ_m:	Mean stress invariant
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$:	Cartezian stress tensor
φ:	Friction angle
φ_f:	Factored friction angle
ψ:	Dilation angle
$\varepsilon^{\bullet VP}$:	Visco plastic strain rate
$\bar{\sigma}$:	Deviator stress

Appendix 1

Plastic Potential Derivatives

$$M^1 = \frac{1}{3\left(\sigma_x+\sigma_y+\sigma_z\right)}\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

$$M^2 = \frac{1}{3}\begin{bmatrix}2&-1&-1&0&0&0\\-1&2&-1&0&0&0\\-1&-1&2&0&0&0\\0&0&0&6&0&0\\0&0&0&0&6&0\\0&0&0&0&0&6\end{bmatrix} \qquad \sigma=\begin{Bmatrix}\sigma_x\\\sigma_y\\\sigma_z\\\tau_{xy}\\\tau_{yz}\\\tau_{zx}\end{Bmatrix}$$

$$M^3 = \frac{1}{3}\begin{bmatrix}s_x&s_z&s_y&\tau_{xy}&-2\tau_{xy}&\tau_{zx}\\&s_y&s_x&\tau_{xy}&\tau_{yz}&-2\tau_{zx}\\&&s_z&-2\tau_{xy}&\tau_{yz}&\tau_{zx}\\&&&-3s_z&3\tau_{zx}&3\tau_{yz}\\&&&&-3s_x&3\tau_{xy}\\&&&&&-3s_y\end{bmatrix}$$

$$DQ_1 = \sin\varphi$$

$$DQ2=\frac{\cos\theta}{\sqrt{2}\,t}\left[1+\tan\theta\tan 3\theta+\frac{\sin\varphi}{\sqrt{3}}(\tan 3\theta-\tan\theta)\right]$$

$$DQ3=\frac{\sqrt{3}\sin\theta+\sin\varphi\cos\theta}{t^2\cos 3\theta}$$

where:

t = Second deviatoric stress invariant.

θ = Lode angle.

ψ = Dilation angle.