Complementary and Single Variational Principles for Investigating Harmonic Eddy-Current in a Thick Sheet

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Abstract

Investigation of eddy-current problems through a thick sheet is presented based on the development of a complementary formulation for electric and magnetic field intensities. These two types of formulation (systems) can be solved independently and the two solutions complement each other.

A comparison is made using a single variational principle, which involves both complementary systems simultaneously. Accurate results are given for estimating the resistance and inductance through which the eddy current can be estimated.

الخلاصـــــــة

يقدم البحث طريقه تحليليه للتيارات ألدوامة في الأوساط السميكة معتمدا على منظومتي الحل الكهربائية و المغناطيسية و التي يمكن أن تستخدم بشكل منفرد أو متمم أحدهما للآخر.

أعطت ألنتائج قيم دقيقة لحساب معالم الدائرة ألمكافئة للوسط باستخدام كلا ألحلين و التي تمثل الأسلوب ألأمثل لمعرفة ألتيارات ألدوامة.

1. Introduction

A characteristic of magnetic materials that is very significant in the energy efficiency of an electromagnetic device is the energy loss within the magnetic material itself. The theoretical description of the basic mechanism that results in magnetic material losses can be given as a simple explanation where the energy is used to effect magnetic domain wall motion as the domains grow and rotate under the influence of an externally applied magnetic field. When the external field is reduced or reversed from a given value, domain wall motion is irreversible and manifests itself as heat within the magnetic material. The rate at which the external field is changed has a strong influence on the magnitude of the loss, and the loss is generally affected by the frequency of the variation of the magnetic field and the electric conductivity of the magnetic material.

Eddy current losses are caused by induced electric currents; they tend to flow in closed paths within the magnetic material itself. The eddy current is roughly proportional to the square of the lamination thickness and inversely proportional to the electric resistivity of the material, an empirical relationship can be expressed for this eddy current loss which proportional to the square for both the frequency and the maximum value of the magnetic flux density [1].

Different efforts using analytical and numerical methods were focused for estimating eddy current losses for bounding problems [2-5]. Other approaches give an estimating value for eddy current problems using the complementary functional for bounding formulation problems [6-9].

In this work, a complementary formulation for non-bounding problems and the solution with error-based approach is given. This formulation includes the E-system and the H-system as an independently solvable complementary variational principle to be suitable for thick sheet analysis. A single variational principle that involves both complementary systems simultaneously is presented too. Computational results for the resistance and inductance and the energy loss associated with are demonstrated using both techniques, the single and the complementary solutions.

2. Formulation of the Problem

The considered problem is a thick conducting sheet oriented as shown in **Fig.(1)**. It has a total thickness of 2b in the y-direction and extends indefinitely along the x-and z-axes. It carries a specified surface current density k per unit width flowing in the z-direction. There are two thin trips at the top and bottom of the sheet with thickness (a) due to the effect of high frequency; hence all fields are largely confined to these two thin trips.

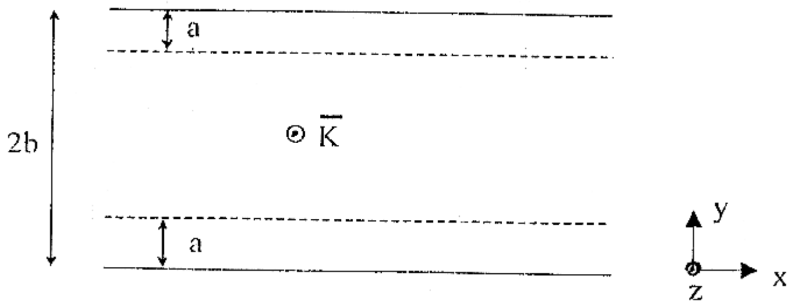

Figure (1) Geometry of the problem

For harmonic (sinusoidal) variation fields, Maxwell's equations can be expressed as:

$$\mathbf{Curl}\,\overline{\mathbf{H}} = \nabla \times \overline{\mathbf{H}} = \overline{\mathbf{J}} \quad \text{..... (1)}$$

$$\mathbf{Curl}\,\overline{\mathbf{E}} = \nabla \times \overline{\mathbf{E}} = -\mathbf{j}\varpi\overline{\mathbf{B}} \quad \text{..... (2)}$$

Constitutive relationships relating the electric and magnetic systems are:

$\overline{\mathbf{B}} = \mu\overline{\mathbf{H}}$, $\overline{\mathbf{J}} = \sigma\overline{\mathbf{E}}$

and $\overline{\mathbf{H}} = \upsilon\overline{\mathbf{B}}$, $\overline{\mathbf{E}} = \rho\overline{\mathbf{J}}$

σ : Conductivity of the sheet material,
μ : Permeability, and
ϖ : angular frequency.
$\overline{\mathbf{E}}, \overline{\mathbf{H}}$: Electric and magnetic field intensities,
$\overline{\mathbf{B}}$: Magnetic flux density,
$\overline{\mathbf{J}}$: Current density.

Due to the problem geometry, the fields vary with y only and each field has only one component, so the resulting components are;

$\overline{\mathbf{H}} = \mathbf{H(y)}\overline{\mathbf{a}}_{\mathbf{X}}$, $\overline{\mathbf{E}} = \mathbf{E(y)}\overline{\mathbf{a}}_{\mathbf{Z}}$, and $\overline{\mathbf{J}} = \mathbf{J(y)}\overline{\mathbf{a}}_{\mathbf{Z}}$, with the aid of equations (1) and (2) we get:

$$-\frac{\mathbf{dH}}{\mathbf{dy}} = \mathbf{J} \quad \text{..... (3)}$$

$$\text{and } \frac{\mathbf{dE}}{\mathbf{dy}} = -\mathbf{j}\varpi\mathbf{B} \quad \text{..... (4)}$$

Using the boundary conditions: $n \times E = 0$ and $n \times H = 0$ at the front and rear surfaces, and $n \times H$ at the top and bottom surfaces is constant on each, then we get.

$$H(-b) = -H(b) = \frac{k}{2} \quad \ldots\ldots (5)$$

This relation belongs to the H-system where the H-system contains H, J and equations (1) and (3) while the E-system contains E, B and equations (2) and (4).

H(y) and E(y) can be represented as polynomials in *y* using simple numerical solutions [1],

$$H(y) = \left[P_1 y + P_3 y^3 + P_5 y^5\right] + j\left[q_1 y + q_3 y^3 + q_5 y^5\right] \quad \ldots\ldots (6)$$

$$E(y) = \left[r_O + r_2 y^2 + r_4 y^4\right] + j\left[s_O + s_2 y^2 + s_4 y^4\right] \quad \ldots\ldots (7)$$

Using equations (6) and (7) in equations (3) and (4) to get J and B for the H-system and the E-system respectively

$$J(y) = -\left[P_1 + 3P_3 y^2 + 5P_5 y^4\right] - j\left[q_1 + 3q_3 y^2 + 5q_5 y^4\right] \quad \ldots\ldots (8)$$

and:

$$B(y) = -\frac{1}{\varpi}\left[2r_2 y + 4r_4 y^3\right] + \frac{j}{\varpi}\left[2s_2 y + 4s_4 y^3\right] \quad \ldots\ldots (9)$$

Apply equation (5) for the H-system then:

$$-\left[P_1 b + P_3 b^3 + P_5 b^5\right] - j\left[q_1 b + q_3 b^3 + q_5 b^5\right] = \frac{k}{2} \quad \ldots\ldots (10)$$

Equation (8), (9) and (10) can be used for the independent H-system and E-system.

To get the complementary variational principles we can use the variational principle in its general in its general relation as:

$\delta\lambda = 0$, where: λ is the complex phasor associated with the sinusoidal component.

This principle can be split into complementary variational principles as:

$$\lambda(H,E) = \frac{1}{2}\left\{\iiint\left[\mu\bar{H}.\bar{H} + \frac{1}{j\varpi}\rho\bar{J}.\bar{J}\right]dV + \iiint\left[\upsilon\bar{B}.\bar{B} + \frac{1}{j\varpi}\sigma\bar{E}.\bar{E}\right]dV\right\}$$
$$-\frac{1}{j\varpi}\iint\left[\bar{n}\times\bar{H}.\bar{E}\right]ds \quad \ldots\ldots (11)$$

So $\lambda(H,E) = k_1(H) + k_2(E)$ where:

$$k_1(H) = \frac{1}{2}\iiint\left[\mu\overline{H}.\overline{H} + \frac{1}{j\varpi}\rho\overline{J}.\overline{J}\right]dV \quad \text{.......... (12)}$$

and:

$$k_2(E) = \frac{1}{2}\iiint\left[\upsilon\overline{B}.\overline{B} + \frac{1}{j\varpi}\sigma\overline{E}.\overline{E}\right]dV - \frac{1}{jw}\iint\left[n \times \overline{H}.\overline{E}\right]ds \quad \text{.......... (13)}$$

k_1 and k_2 are complementary functional. All the quantities $H, E, B, J, \lambda, k_1, k_2$ are complex phasors. Therefore:

$k_1(H) = 0$ and, $k_2(E) = 0$

These are the complementary variational principles, which can be solved independently of each other.

3. Complementary Solutions

In this section, the complementary solutions will be described in terms of the estimated values of resistance and inductance. The exact solutions of R and L are given as [10,11]:

$$R = \frac{1}{2\sigma\Delta}\left[\frac{\sinh\gamma + \sin\gamma}{\cosh\gamma - \cos\gamma}\right] \quad \text{.......... (14)}$$

and:

$$L = \frac{1}{2\sigma\varpi\Delta}\left[\frac{\sinh\gamma - \sin\gamma}{\cosh\gamma - \cos\gamma}\right] \quad \text{.......... (15)}$$

where: $\gamma = \frac{2b}{\Delta}$ and $\Delta = \sqrt{\pi f \sigma \mu}$ =Depth of penetration. These values can be compared with the estimated values in the complementary systems where the resistance represented as:

$$R(H) = \iiint\frac{\rho\overline{J}.\overline{J}^*}{A}dV \qquad \text{For the H-system .... (16)}$$

$$R(E) = \iiint\frac{\sigma\overline{E}.\overline{E}^*}{A}dV \qquad \text{For the E-system .... (17)}$$

While the inductance is given by:

$$L(H) = \iiint\frac{\mu\overline{H}.\overline{H}^*}{A}dV \quad \text{and,} \quad L(E) = \iiint\frac{\upsilon\overline{B}.\overline{B}^*}{A}dV \quad \text{and,}$$

$$\mathbf{A} = |\mathbf{I}|^2 \quad \text{.......................................} \quad (18)$$

at the exact solution $R(H) = R(E) = R$ exact, and $L(H) = L(E) = L$ exact.

All the numerical formulation of R and L can be evaluated once the parameters, P, q, r, s are obtained from the solution.

Using equations (14-18) with equations (6-9) then a resulting numerical and exact value of resistance and inductance can be estimated at different values of sheet thickness and for any order of polynomial for both systems.

4. Thick Sheet

At high values of the ratio (b/Δ), the exact values for resistance and inductance approach are respectively, direct and inverse proportionality to (b/Δ). Using the exact relations for R and L, (equations (14, 15)) it is found that, at $(b/\Delta)\rangle\rangle 1$ then:

$$\mathbf{R = (b/\Delta)R_{dc}} \qquad \text{where:} \qquad \mathbf{R_{dc} = \frac{1}{2b\sigma}}$$

and $$\mathbf{L = [1.5/(b/\Delta)]L_{dc}} \qquad \text{where:} \qquad \mathbf{L_{dc} = \mu b/6}$$

The high ratio is obtained from a thick sheet or, equivalently, at high frequency. The current, and hence all fields are largely confined to two thin strips at the top and bottom of the sheet. The trial function equations (6-10) should be modified at high values of (b/Δ). The modified trial functions to Fig. 1 as follows [1]:

$$\mathbf{H(y) = \left[\left(P_O + P_1 y + P_2 y^2 + P_3 y^3\right) + j\left(q_O + q_1 y + q_2 y^2 + q_3 y^3\right)\right] \quad 0 \le y \le a}$$
$$\mathbf{H(y) = 0 \qquad a \le y \le b} \quad \text{......} \ (19)$$

$$\mathbf{E(y) = \left[\left(r_O + r_1 y + r_2 y^2 + r_3 y^3\right) + j\left(s_O + s_1 y + s_2 y^2 + s_3 y^3\right)\right] \quad 0 \le y \le a}$$
$$\mathbf{E(y) = 0 \qquad a \le y \le b} \quad \text{......} \ (20)$$

Equation (19) and (20) apply in the lower half of the sheet, their counter parts in the upper half should make (E) symmetric and (H) anti symmetric about the mid plane. The trial functions for (J) and (B) are obtained as:

$$\mathbf{J(y) = \left[-\left(P_1 + 2P_2 y + 3P_3 y^2\right) - j\left(q_1 + 2q_2 y + 3q_3 y^2\right)\right] \quad 0 \le y \le a}$$
$$\mathbf{J(y) = 0 \qquad a \le y \le b} \quad \text{..........} \ (21)$$

and:
$$B(y)=\left[-\frac{1}{\varpi}\left(s_1+2s_2y+3s_3y^2\right)+\frac{j}{\varpi}\left(r_1+2r_2y+3r_3y^2\right)\right] \quad 0\le y\le a$$
$$B(y)=0 \quad a\le y\le b \qquad \text{.......... (22)}$$

The H-system trial functions involve eight parameters (P_O-P_3, q_O-q_3).

The boundary condition at the bottom surface $H(0)=k/2$ reduces the unknown parameters to 6. Moreover, at (y = *a*) we must have $H(a)=0$ and $J(a)=0$ which leaves only two unknown parameters to be determined from the *H*-system solution.

The E-system trial functions also start with eight parameters (r_O-r_3, s_O-s_3), the conditions at y = *a* are, $E(a)=0$ and $B(a)=0$ reduces the unknown parameters by 4 to be determined for the E-system solution.

The sheet problem was solved using the complementary formulations of equations (11-13) with the modified trial functions described by equations (19-22).

Figures (2) and **(3)** show the resulting resistance and inductance estimates obtained using different values of strip thickness (a). The curves indicate that the best choice for *a* is around (2.8Δ), both complementary estimates for R and L almost coincide with the exact value.

However the curves also show that beyond say $a=\Delta$ the averages of complementary estimates lie consistently very close to the exact values; that is, the value chosen for *a* is not critical if, rather than either estimate by itself, the average of complementary estimates is used.

The complementary estimates for both *R* and *L* are non-bounding. However, resistance estimates exhibit apparent bounded ness beyond $a=1.4\Delta$.

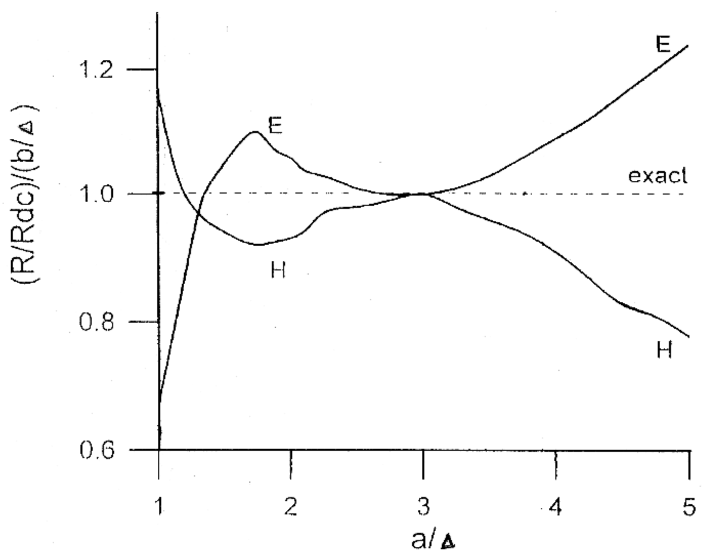

Figure (2) Complementary resistance estimates for thick sheet approximation

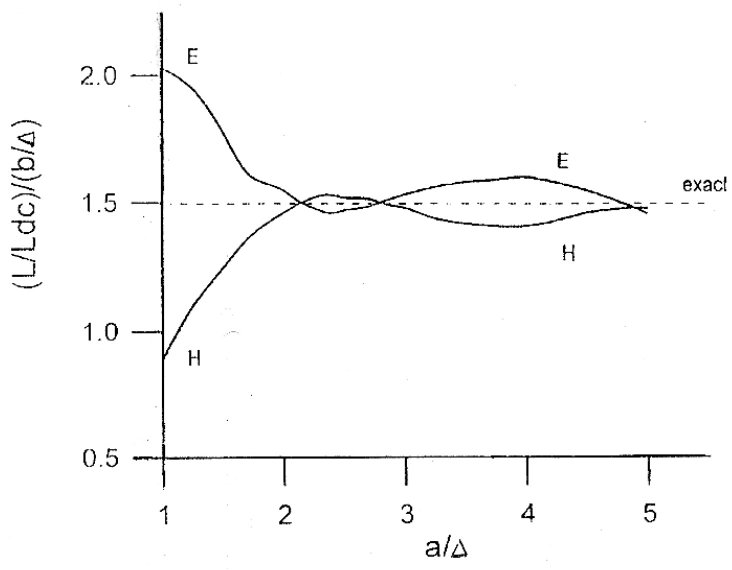

Figure (3) Complementary inductance estimates for thick sheet approximation

5. A Single Solution

The results of the last section was obtained using the complementary formulations (equation 11) which evolve from the variational principle $(\delta\lambda(\text{real})=0, \delta\lambda(\text{imaginary})=0, \text{and}\ \delta\lambda(\text{complex})=0)$ corresponding to the complex double frequency Ligurian λ.

The error based derivation also produces the variational principle of $\delta\lambda_O = 0$ which corresponds to the real time-invariant Ligurian λ_O [12]. Minimization of λ_O provides a valid solution formulation, although it does not split into complementary formulations.

In this section we present the results obtained using the variational principle $\delta\lambda_O = 0$ with the trial variables of equations (6-9). The solution involves the variables of both complementary systems simultaneously; that is, minimization of λ_O describes a single solution for all unknown parameters of both H-and E-systems.

It will be recalled that the multiplier α and the integration interval τ in the general total Ligurian relation, $\lambda(t)=\lambda_m(t)+\alpha\int_{t-\tau}^{t}\lambda_C(t)dt$, may be assigned arbitrary positive values, in the above relation λ_m and λ_c are the magnetic and conduction constitutive errors or Ligurians.

The product of $\alpha\tau$ is acts as a weighting factor that controls the accuracy of the solution. Assigning it a large value causes the solution to cater for the conduction constitutive relationship. Conversely, assigning $\alpha\tau$ a small value causes the solution to cater for the magnetic constitutive relationship. For a given problem, there may be an optimum choice for $\alpha\tau$, the value of $\alpha = 1$ and $\tau = [\pi/2\varpi]$ can be used for good expectation. One unweighted error measure is the difference between complementary estimates of complex energies, $k_1(H)+k_2(E)$, where k_1 and k_2 are defined in equations (12, 13). **Figure (4)** shows the variation of this error with $\alpha\tau$ at two values of (b/Δ). It is clear from the figure that $\alpha\tau$ can be increased to extreme values, at the same time error is reduced at high $\alpha\tau$. Also within a wide acceptable range, the value chosen for $\alpha\tau$ is not critical. **Figure (4)** also shows that over a wide range, the accuracy of the single solution is better than of the complementary solution.

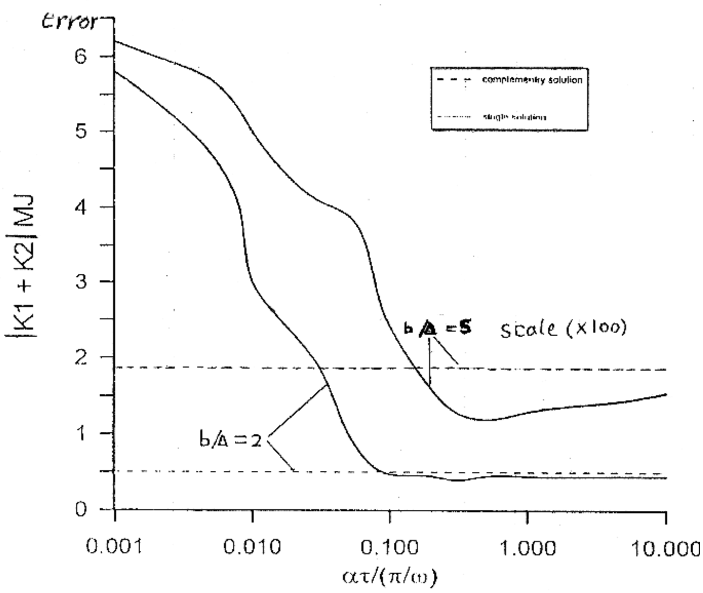

Figure (4) Effect of ατ on the overall accuracy of the single solution

Figure (5) compares Ligurian error λ_c with complementary functionless k_1, k_2, there is improvement in accuracy holds even at small values of (b/Δ). This is interesting because it implies that the independent complementary solutions do not yield the best values for the unknown parameters in the trial functions, which is in keeping with the fact that complementary variational principles are merely stationary, and not external. **Figure (6)** and **(7)** display resistance and inductance estimates obtained from the single solution at the chosen value of $\alpha\tau$. The curves have fewer crossovers, and in general behave more uniformly than those obtained from independent complementary solutions behave. In sharp contrast, it is the inductance curves now, which exhibit apparent bounded behavior, although the resistance curves also exhibit such behavior up to (b/Δ) just under three.

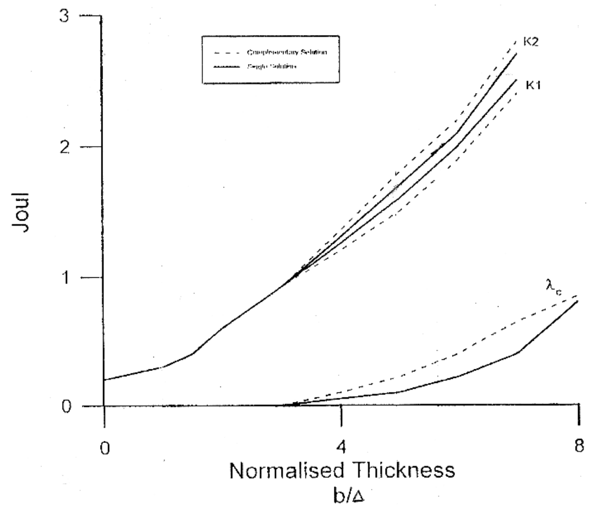

Figure (5) Complementary functionals and ligurian errors at $\alpha\tau=\pi/2\varpi$

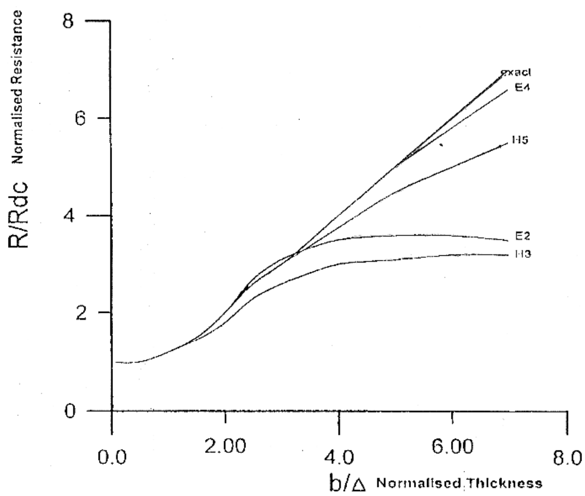

Figure (6) Resistance estimates from single solution at $\alpha\tau=\pi/2\varpi$

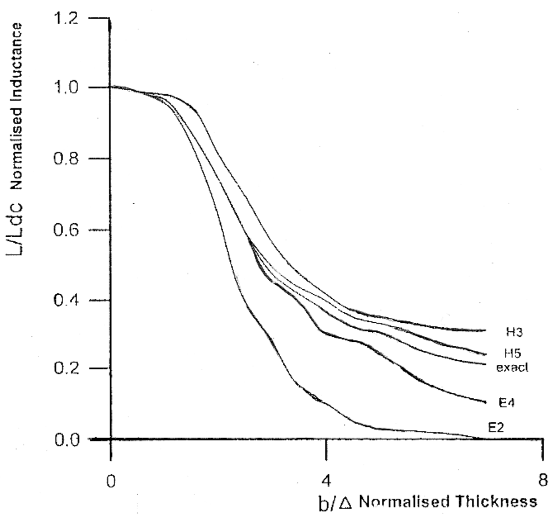

Figure (7) Inductance estimates from single solution at $\alpha\tau=\pi/2\varpi$

6. Conclusions

A numerical solution is presented based on complementary formulations for solving harmonic eddy current problem in a thick sheet. The two sets *E*-system and *H*-system of solutions complement each other about the exact solution. A good accuracy was achieved using Ligurian error approach.

The associated results are obtained by either of the complementary solutions alone or by performing both complementary solutions. In addition, a single solution that involves both system variables simultaneously was considered which give better accuracy. This solution has the advantages of both complementary solutions plus improved accuracy, but suffers from the economical disadvantage of handling double-sized matrices. It is clear from results obtained the H-system estimates are more accurate than the E-system estimates because the boundary conditions of equation (5) which force the solution belong to the H-system and are enforced explicitly on H-system variables equation (10) they enter the E-solution only naturally through the last term in equation (13).

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