Non-Linear Analysis of Beams on Elastic Foundation by Finite Element Method

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Abstract

This study is concerned with the behavior of beams on elastic foundation using finite element methods. The nonlinear behavior for reinforced concrete was taken into account addition to the nonlinear contact behavior of elastic foundation. As known, the elastic foundation cannot carry the tensile contact pressure, this pressure will be ignored when it occurs and then substituting zero value for the spring constant. It was found that the nonlinear behavior of materials would have low effect on deflection when compared with linear behavior. In contrast, the nonlinear contact behavior of elastic foundation would give deflection values greater than the linear behavior.

A computer program (written by the researcher) is used for this purpose, where the cross-section properties can be changed such as different thicknesses and reliable results are obtained. Moreover, in the case of exiting tensile contact pressure between the surfaces and using the linear material behavior, the deflection with load will be nonlinear.

الخلاصــــة

تتعلق هذه الدراسة بمسألة التصرف الاخطي العتبات المسنده على أسس مرنه بطريقه العناصر المحددة . تم اخذ التصرف اللاخطي للخرسانة المسلحه بنظر الاعتبار كما اخذ التصرف اللاخطي للألتصاق للأساس المرن بنظر الاعتبار أيضا. من المعروف أن الأساس المرن لا يتحمل إجهاد شد لذلك هذا الإجهاد سوف يهمل ويعوض عن قيمه الثابت بصفر. لقد وجد إن التصرف اللأخطي للمواد يعطي تأثير منخفض على التشوه عندما يقارن مع التصرف الخطي كما ان التصرف اللأخطي للألتصاق للأساس المرن يعطي تشوه أعلى من التصرف الخطي.

برنامج حاسوبي (مكتوب من قبل الباحث) استخدم لهذا الغرض حيث ان خواص المقطع يمكن تغيرها مثل السمك حيث ان النتائج كانت منطقيه.كما أنه في حاله وجود شد بين السطحين و باستعمال التصرف الخطي للمواد يكون التشوه مع الحمل لاخطي.

1. Introduction

Sometimes a structure is supported by another, but analysis is required for only the first of the two. Then it suffices to model the effect of the second structure on the first. It is not needed to model the second structure in such details so that stresses within it can be determined. Examples include a rail on a roadbed or a pavement slab on soil. The rail or slab must be analyzed; the supporting effect of the roadbed or soil must be modeled [1]. It was observed that for beams on elastic foundation the linear elastic analysis will yield tensile as well as compressive contact pressures. Tensile contact pressure can also result from local uplifting forces due to wind load [2]. The finite element method has been used to solve the problems of two-dimensional beam-column (under axial and bending action) on a nonlinear elastic foundation only [3&4].

The basic differential equation for beams on elastic foundation is:

$$EI\frac{d^4y}{dx^4} - Ks\,y = q \qquad \textbf{(1)}$$

where:
E= modulus of elasticity of the beam (kN/m^2),
I= moment of inertia of the beam (m^4),
Ks= modulus of subgrade reaction (kN/m^2),
q= laterally distributed load on beam (kN/m), and,
y= deflection of the beam (m).

2. Finite Element Solution

The finite element method is efficient for solving a beam on elastic foundation. Based on Eq.(1), it is easy to account for boundary conditions, beam weight, material nonlinearity, and nonlinear soil effects [5].

The beam is divided into elements, as shown in **Fig.(1)**, vectors which represent the axial and bending displacements are {u} and {v}, respectively:

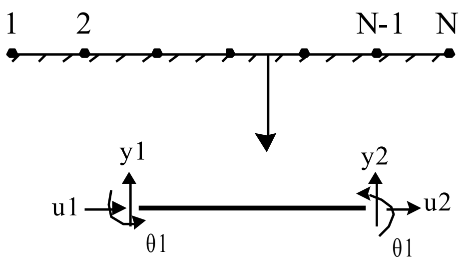


Figure (1) Displacement Component of Beam Element with Six Degree of Freedom

$$\{u\}=\lfloor u1 \quad u2 \rfloor^T$$
$$\{v\}=\lfloor y1 \quad \theta 1 \quad y2 \quad \theta 2 \rfloor^T \qquad \text{.........} \textbf{(2)}$$

These displacement components can be assembled in on column vector {y}:

$$\{y\}=\lfloor u1 \quad y1 \quad \theta 1 \quad u2 \quad y2 \quad \theta 2 \rfloor^T \qquad \text{.........} \textbf{(3)}$$

Let Uo(x) and Vo(x) be the axial and bending displacements at any point along X-axis, respectively, then:

$$Uo(x) = Na\{u\}$$
$$Vo(x) = Nb\{v\} \qquad \text{.........} \textbf{(4)}$$

where, Na is the shape functions defining a linear interpolation of Uo(x) between nodes, and Nb comprises the cubic beam function of interpolation polynomial [6],

$$Na = \lfloor N1 \quad N2 \rfloor^T$$
$$Nb = \lfloor N3 \quad N4 \quad N5 \quad N6 \rfloor^T \qquad \text{.........} \textbf{(5)}$$

where:

$$N1 = 1 - \frac{x}{L} \qquad N2 = \frac{x}{L}$$

$$N3 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad N4 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$N5 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \qquad N6 = \frac{x^3}{L^2} - \frac{x^2}{L}$$

The stiffness matrix of a beam element in bending can be represented by the following equation:

$$[K] = \int_0^L \lfloor B \rfloor^T EI \lfloor B \rfloor dx \qquad \text{.........} \textbf{(6)}$$

where $\lfloor B \rfloor = \frac{d^2 \lfloor N \rfloor}{dx^2}$, for bending and for constant EI, the element stiffness matrix for beam element can be found in Reference [5&6].

Similar approach can be used for steel reinforcement along the axis of beam. The stiffness matrix of bars can be represented below,

$$[K]_{sr} = E_{sr} A_{sr} \begin{bmatrix} \frac{1}{L} & 0 & \frac{-dy}{L} & \frac{-1}{L} & 0 & \frac{dy}{L} \\ & \frac{12dy^2}{L^3} & \frac{6dy^2}{L^2} & 0 & \frac{-12dy^2}{L^3} & \frac{6dy^2}{L^6} \\ & & \frac{4dy^2}{L} & \frac{dy}{L} & \frac{-6dy^2}{L^2} & \frac{2dy^2}{L} \\ & & & \frac{1}{L} & 0 & \frac{-dy}{L} \\ & & & & \frac{12dy^2}{L^3} & \frac{-6dy^2}{L^2} \\ & & & & & \frac{4dy^2}{L} \end{bmatrix}$$

where E_{sr} and A_{sr} are the modulus of elasticity and area of steel bars; dy distance from concrete neutral axis to steel reinforcement as shown in **Fig.(2)**.

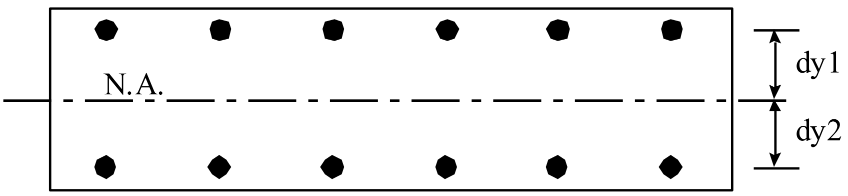


Figure (2) Cross Section of Reinforced Concrete Beam

The stiffness matrix of Winkler foundation model is [5]**:**

$$[K_w] = \int_0^L \lfloor N \rfloor^T Ks \lfloor N \rfloor dx \quad \textbf{.. (7)}$$

$$[K_w] = \frac{Ks * L}{420} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L \\ & & & 0 & 0 & 0 \\ & & & & 156 & -22L \\ & & & & & 4L^2 \end{bmatrix}$$

3. Material Constitutive Relationships

Concrete; for concrete in compression, the model for the stress-strain relationship proposed in BS 8110 [7] as shown in **Fig.(3-A)** is used, the ultimate compressive strain, ε_{cu} is limited to 0.0035, the curved portion of the stress-strain curve is defined by:

$$\sigma = 5500\sqrt{\sigma_{cu}}\,\varepsilon - 11.3*10^{6}\varepsilon^{2} \quad \textbf{(8)}$$

with $\varepsilon_o = 2.44*10^{-4}\sqrt{\sigma_{cu}}$, and the initial modulus of elasticity is:

$$Ei = 5500\sqrt{\sigma_{cu}} \quad \textbf{(9)}$$

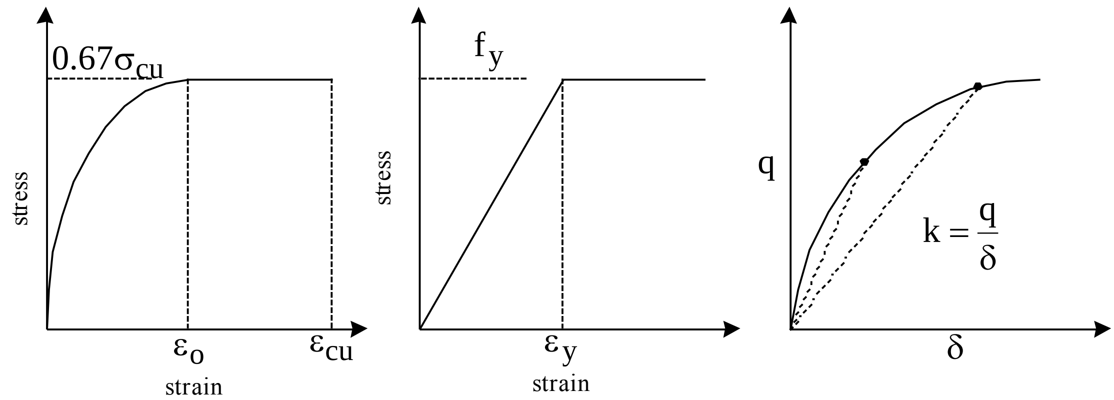
in which σ_{cu} is the concrete cube strength in MPa.

The tensile strength of concrete is relatively low so that, concrete is assumed incapable to resist any tension.

Steel Reinforcement; A bilinear stress-strain curve is adopted for this type of steel as shown in **Fig.(3-B)**. In this stress-strain curve, equal yield stress, fy, in tension and compression is assumed.

Modulus of Subgrade reaction; the modulus is a conceptual relationship between soil pressure and deflection. **Figure (3-C)** shows the relation between pressure of plate and deflection of this plate, the basic equation when using plate-load test data is [1&8]**:**

$$Ks = \frac{q}{\delta} \quad \textbf{(10)}$$



A **B** **C**

Figure (3) A-Stress-Strain Curve for Concrete
B-Bilinear Stress-Strain Curve for Steel Reinforcement
C-Load-Settlement Curve for Plate Bearing Test

In the case of linear material analysis and when there are separations of contact surfaces with tensile contact pressures, the following steps are used [2]:

1. A linear elastic solution is obtained.
2. If all the contact pressures are compressive the problem is terminated. If otherwise, proceed to next step.
1. Find out the nodes which are associated with tensile or zero contact pressures and make the corresponding rows and columns in the original stiffness matrix zero.
2. Invert the new stiffness matrix and repeat step No.1.

4. Convergence Criterion

The nonlinear algebraic equations can be solved iteratively, as illustrated in **Fig.(4)** in which R and d denote a representative load and displacement respectively.

For the first stage of solution, the material properties are assumed constant and a set of nodel displacements corresponding to a specified applied loading is determined. From these displacements, strains throughout the beam are determined, which are used to define the secant values of material properties for the second stage of the solution. The process is repeated until the calculated displacements have converged.

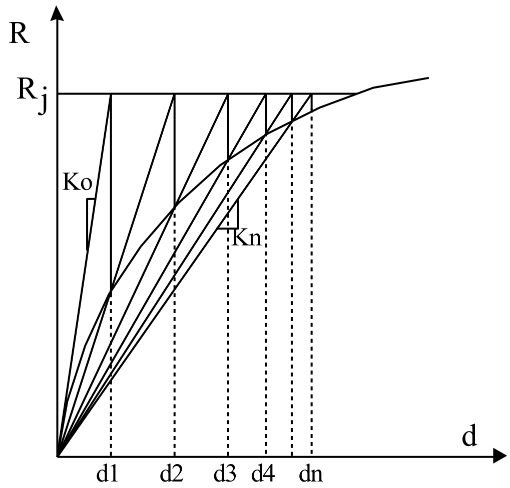
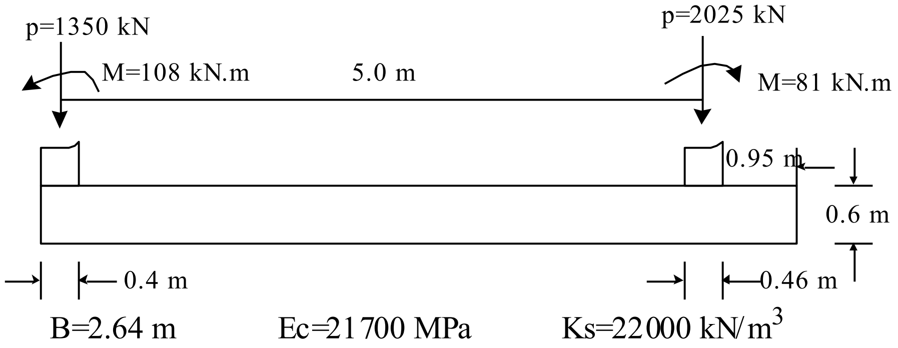


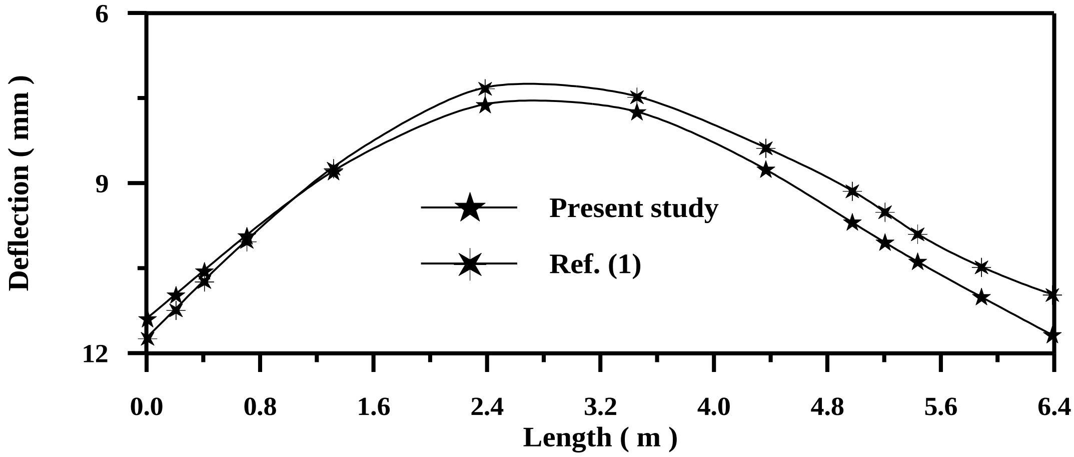
Figure (4) Solution Procedure in a Nonlinear Problem (Secant Method)

5. Numerical Examples

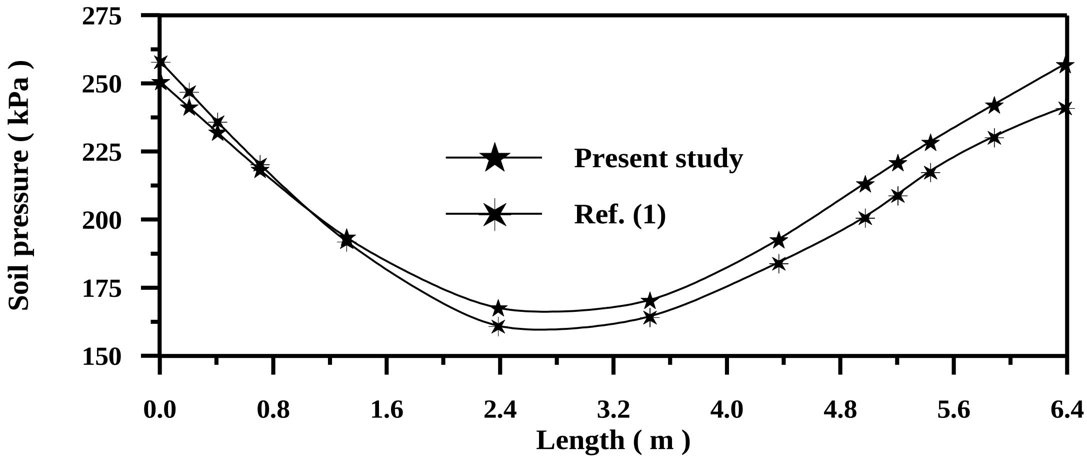
Linear Analysis: The results are compared with those obtained by Bowles [1], the general footing data are shown in **Fig.(5-A)**. The computer program gives good results, but the computing results are greater than the reference results after mid-footing to the free edge, while before mid-footing the computing and the reference results are very close. Maximum difference between the computed values and the reference values in **Fig.(5-B)** is approximately 0.714 mm and occurs at the free edge, while for all footing the average difference is equal to 0.23 mm. In addition, the maximum difference between the computed values and the reference values in **Fig.(5-C)** is approximately 15.7 kPa and occurs at the free edge, while for all footing the average difference is equal to 5.05 kPa.



(A)



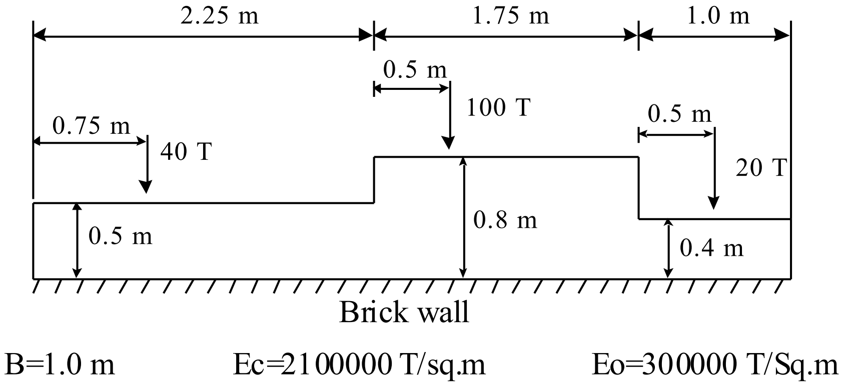
(B)



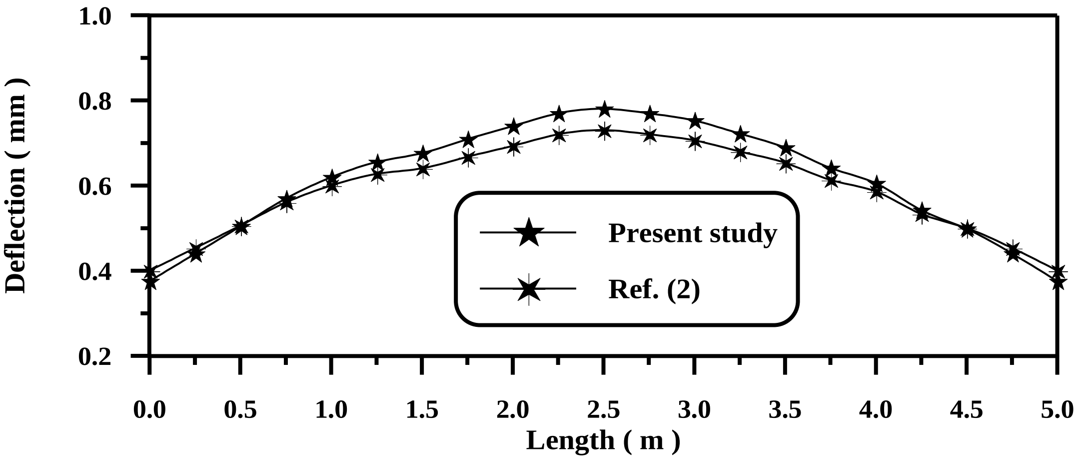
(C)

Figure (5) A-Foundation Detail Analyzed by Ref. (1)
B & C-Comparison of Results of Present Study with Reference Values

The ability of the computer program was tested with variable beam thickness. **Figure (6-A)** shows the variable beam thickness in details and by the comparison of results of the present study with the reference values, the computer program gives good results. The maximum difference occurs at the center of the beam and is approximately equal to 0.05 mm, and the percent of this difference is equal to 7%. While in the other locations of the span this percent is very small (about 1%).



(A)



(B)

Figure (6) A-Beam Detail as Analyzed by Ref. (2)
B-Comparison of Results of Present Study with Reference Values

Nonlinear Analysis: The ability of the computer program was tested with nonlinear elastic foundation only. **Figure (7)** shows the beam under constant uniform distributed load (downward) and central concentrated force (upward, increased with test). When P is small the contact pressures between soil and beam are compression. But, after the increase of P, tensile contact pressure will occur at the center of the beam, therefore the deflection must be nonlinear with load. Maximum difference occurs at initial test (P=0) but in the final stages there are small differences.

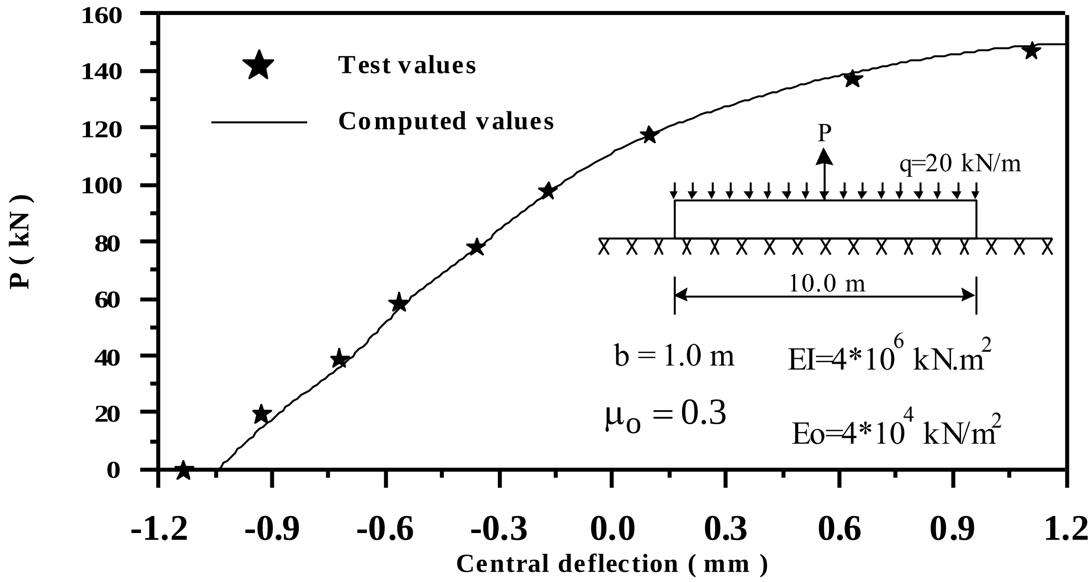


Figure (7) Comparison of Results of Present Study with Test Values [2]

Figure (8-A) shows details of a uniform reinforced concrete beam under central force, the ultimate compressive strength of concrete is 20 MPa (cube) and the initial modulus of elasticity of concrete is equal to 24597 MPa. The modulus of elasticity of steel reinforcement is 200000 MPa and the yield stress f_y is equal to 300 MPa. **Figure (8-B)** shows the variation of pressure of the bearing plate (0.3m * 0.3m) with deflection of this plate. **Figure (8-C)** shows the variation of maximum deflection (under point load) with load. The effect of nonlinear soil only is very clear with increasing the load, the increasing percent in deflection is 104% (at 210 kN) due to nonlinear soil only. But the effect of nonlinear material only (concrete and steel) is very low and the increasing percent in deflection is 1.87% (at 210 kN) due to nonlinear materials.

For nonlinear materials, the difference between the maximum deflection (under point load) and minimum deflection (at free edges) is 2.0795 mm at load equal to 210 kN as shown in **Fig.(8-D)**. This gives low curvature (low axial strains or bending strains), and this is the reason of low effect of material nonlinearity (approximate elastic state because of low strain). Therefore in this type of structures the ultimate load depends only on the capacity of the elastic foundation. The effect of steel reinforcement can be neglected because bending cracks are very little and the effect of this crack is very low for this type of load.

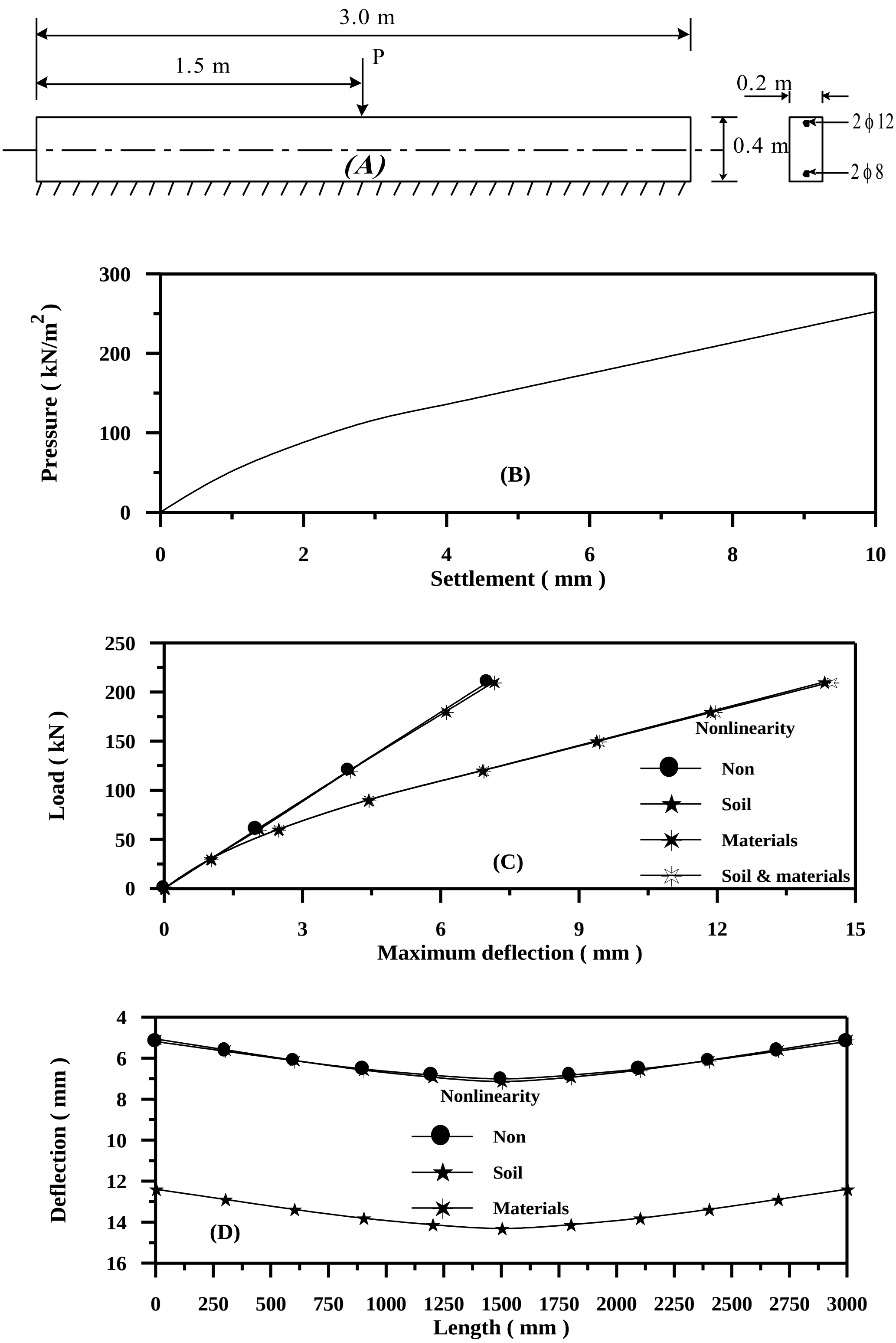


Figure (8) A-Beam Detail
B-Pressure-Settlement Curve for Plate Bearing Test
C-Load-Deflection Curve for Linear and Nonlinear Behavior
D-Deflected Shape at Load P=210 kN for Linear and Nonlinear Behavior

6. Conclusions

The following points are concluded from the above discussion:-

1. The computed results give good agreement when compared with reference values in linear and nonlinear behavior.

2. The nonlinearity of materials (concrete and steel) behavior gives low deformations effect (deflection) and can be neglected.

3. The nonlinearity of foundation (elastic foundation) behavior gives higher deformations effect (deflection) and the difference between linear and nonlinear is 104% at 210 kN.

4. The effect of tensile contact pressure gives nonlinear relationship for deflection with loads in case of linear material behavior.

5. The effect of steel reinforcement can be neglected.

7. References

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