

Elastic Critical Load of Tapered Members

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Abstract

The elastic critical load for a non-prismatic member is derived using the equations of the modified stability functions for a wide range of tapering ratio having rectangular and square cross sectional shapes bent about the major axis and any other solid cross sectional shape tapered in depth only.

The effective length of members at any support conditions is obtained with respect to the hinged-hinged supports. The elastic critical load is obtained by using the finite elements method as an approximate solution to verify the results.

This study can be used as a reference to obtain the elastic critical load and effective length of columns.

الخلاصة

تم اشتقاق قيمة الحمل الحرج للعتبات اللاموشورية باستخدام المعادلات المحدثّة لدوال الاستقرارية لمدى واسع من معامل التخصر لمقاطع مستطيلة و مربعة تنحني حول المحور الأساسي و أي نوع من المقاطع المتخصرة باتجاه العمق فقط.

تم إيجاد الطول الفعال للعتبات لأي حالة تثبيت بالمقارنة مع حالة التثبيت نوع مفصل-مفصل. إن قيمة الحمل الحرج تم إيجادها كذلك باستخدام طريقة العناصر المحددة كطريقة تقريبية لأغراض التحقق. إن هذه الدراسة يمكن استخدامها كمرجع لإيجاد قيمة الحمل الحرج و الطول الفعال للأعمدة.

1. Introduction

Historically the critical-load formula for elastic buckling was originated with Euler in 1744 ^[1]. Very little additional progress was made for a hundred years, and then in 1845, Lamarle ^[2] pointed out that Euler's formula should be used only for slenderness ratios beyond a certain limit and that experimental data should be relied upon for smaller ratios.

To derive the elastic critical load of columns, this study begins by considering a slender column with pinned ends. This column is loaded axially by axial force Q that is applied through the centroid of the cross section (no eccentricity). When the axial load has a small value, the column remains straight and undergoes only axial compression. This straight form of equilibrium is stable, which means that the column returns to the straight position if it is disturbed. For instance, if a small lateral load is applied which causes the column to bend, the deflection will disappear and the column will return to the original position when the lateral load is removed. As the axial load is gradually increased, it reaches a condition of neutral equilibrium in which the column may have a bent shape. The corresponding value of the load is the critical load. At this load the ideal column may undergo small lateral deflections with no change in the axial force, and a small lateral load will produce a bent shape that does not disappear when the lateral load is removed. To determine the critical load and the deflected shape of the buckled column, one of the differential equations of the deflection curve of a beam is used. This equation should be applicable to a column because, when buckling occurs, bending moments are developed in column, which bends as though it were a beam. Also the second order differential equation in terms of bending moment is used to derive the modified stability functions.

Two conditions, opposite in their effect upon column strength under axial loading, must be considered. If enough axial loads are applied to the columns in unbraced frames dependent entirely on its own bending stiffness for resistance to lateral deflection of the tops of the columns with respect to the bases, the effective length of these columns will exceed the actual length. On the other hand, if the same frame were braced to resist such lateral movement, the effective length would be less than the actual length, due to the restraint provided by the bracing of other lateral support.

2. Modified Slope-Deflection Equations

The modified slope-deflection equations of the column can be arranged in matrix form as given in Equation (1), which is the slope-deflection equation modified by Al-Sarraf ^[5]. These equations are available for evaluating the stability of the non-prismatic columns, subjected to axial load and bending moments as shown in **Fig.(1)**.

$$\begin{Bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{VL} \end{Bmatrix} = \frac{\mathbf{EI}_2}{\mathbf{L}} \begin{bmatrix} \mathbf{S}_1 & \mathbf{SC} & \mathbf{S}_1 + \mathbf{SC} \\ \mathbf{SC} & \mathbf{S}_2 & \mathbf{S}_2 + \mathbf{SC} \\ \mathbf{S}_1 + \mathbf{SC} & \mathbf{S}_2 + \mathbf{SC} & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \frac{\delta}{\mathbf{L}} \end{Bmatrix} \dots\dots\dots (1)$$

The column itself is assumed to be perfectly straight and made of a linear elastic material that follows Hooke's law.

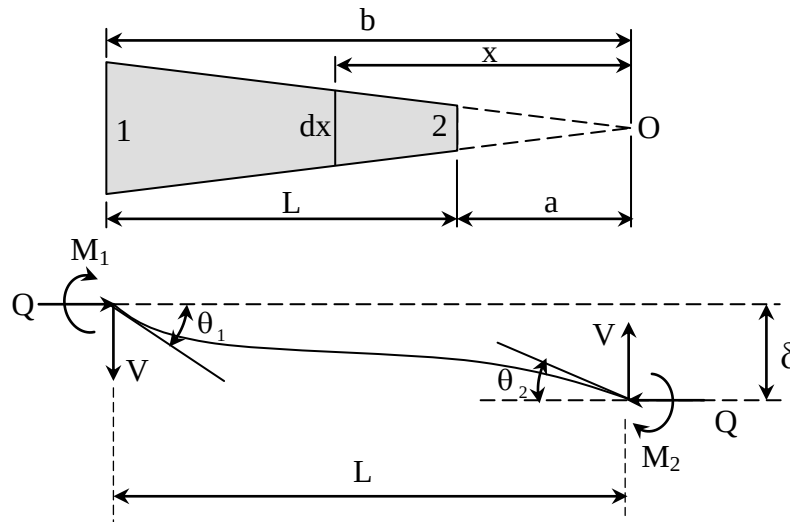


Figure (1) Non-prismatic beam-column element

The above matrix represents the relation between member end forces of the column and the stability functions multiplied by the member ends deformations^[3].

The corresponding equations of the modified stability functions are given below as derived in references (1) and (5) and the values of these equations are tabulated for different tapering ratio as given in Table (A-1) in the Appendix.

$$S_1 = \frac{LZQb^{1.5}}{\omega PEI_2} (\omega Lf_4 + Za^{1.5}) \dots\dots\dots (2)$$

$$SC = \frac{LZQa^{0.5}b}{\omega PEI_2} (\omega Lf_6 + ab^{0.5}Z) \dots\dots\dots (3)$$

$$S_2 = \frac{LZQa^{1.5}}{\omega PEI_2} (\omega Lf_3 + Zb^{1.5}) \dots\dots\dots (4)$$

where:

$$P = Z \left[a \left(\frac{f_5}{b^{-0.5}} + \frac{f_3}{a^{-0.5}} \right) - b \left(\frac{f_4}{b^{-0.5}} - \frac{f_6}{a^{-0.5}} \right) \right] - \omega Lf_1f_2 \dots\dots\dots (5)$$

$$Z = Y_1(\alpha)J_1(\beta) - Y_1(\beta)J_1(\alpha) \dots\dots\dots (6)$$

$$\alpha = \frac{2\omega}{a^{0.5}}, \quad \beta = \frac{2\omega}{b^{0.5}}, \quad \rho_2 = \frac{QL^2}{EI_2\pi^2}, \quad \omega = \left(\frac{a^3Q}{EI_2} \right)^{0.5} \dots\dots\dots (7)$$

$$f_1 = Y_2(\alpha)J_2(\beta) - J_2(\alpha)Y_2(\beta)$$

$$f_2 = Y_1(\alpha)J_1(\beta) - J_1(\alpha)Y_1(\beta)$$

$$f_3 = Y_1(\alpha)J_2(\beta) - J_1(\alpha)Y_2(\beta)$$

$$f_4 = Y_1(\beta)J_2(\alpha) - J_1(\beta)Y_2(\alpha)$$

$$f_5 = J_1(\beta)Y_2(\beta) - Y_1(\beta)J_2(\beta)$$

$$f_6 = Y_1(\alpha)J_2(\alpha) - J_1(\alpha)Y_2(\alpha)$$

$J_1(x)$, $J_2(x)$, $Y_1(x)$ and $Y_2(x)$: are the Bessel functions ^[4] of the first and second kinds.

3. Derivation of the Elastic Critical Load

The elastic critical load of a column with various support conditions can be obtained for different tapering ratios such as a column that is fixed at the base and pinned at the top, fixed at the base and free at the top, fixed at two ends with side-sway, pinned at the base and fixed at the top with side-sway, each case can be determined by using the stiffness matrix of the modified stability functions. The non-dimensional axial load parameter at critical load making the stiffness matrix to vanish is obtained by trial and error with interpolation. The Elastic critical load of the beam-column member mathematically can be obtained by multiplying the non-dimensional axial force parameter with the Euler's load.

A non-prismatic beam-column member, which has length L , cross sectional area A_2 at smaller end, moment of inertia I_2 at smaller end and E is the modulus of elasticity is considered with the properties described below:

3-1 First Model

A beam column member is hinged at two ends, and loaded axially. The elastic critical load is obtained by substituting the boundary condition which is $\delta = 0$, the stiffness matrix becomes as the relation below after omitting the third row and column of the matrix in Equation (1):

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{EI_2}{L} \begin{bmatrix} S_1 & SC \\ SC & S_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \dots\dots\dots (8)$$

The non-dimensional axial load parameter making the stiffness matrix to vanish is obtained by trial and error with interpolation when equating Equation (9) to zero, as given in **Table (1)** for different tapering ratios.

$$S_1 S_2 - SC^2 = 0 \dots\dots\dots (9)$$

3-2 Second Model

A beam column member is fixed from one end and the other is hinged, and loaded axially. The elastic critical load is obtained by substituting the boundary conditions which are

$\theta_2 = 0$ and $\delta = 0$, the stiffness matrix becomes as the relation below after omitting the second and third rows and columns of the matrix in Equation (1):

$$[M_1] = \frac{EI_2}{L} [S_1] \{\theta_1\} \dots\dots\dots (10)$$

The non-dimensional axial load parameter making the stiffness matrix to vanish is obtained by trial and error with interpolation when equating Equation (2) to zero, as given in **Table (1)** for different tapering ratios.

3-3 Third Model

A beam column member is fixed at one end and the other is free, and loaded axially. The elastic critical load is obtained by substituting the boundary condition which is $\theta_2 = 0$, the stiffness matrix becomes as the relation below after omitting the second row and column of the matrix in Equation (1):

$$\begin{Bmatrix} M_1 \\ VL \end{Bmatrix} = \frac{EI_2}{L} \begin{bmatrix} S_1 & S_1 + SC \\ S_1 + SC & A \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \frac{\delta}{L} \end{Bmatrix} \dots\dots\dots (11)$$

The non-dimensional axial load parameter making the stiffness matrix to vanish is obtained by trial and error with interpolation when equating Equation (12) to zero, as given in **Table (1)** for different tapering ratios.

$$S_1 A - (S_1 + SC)^2 = 0 \dots\dots\dots (12)$$

3-4 Fourth Model

A beam column member is fixed at two ends, and loaded axially with side-sway at the top. The elastic critical load is obtained by substituting the boundary conditions which are $\theta_1 = 0$ and $\theta_2 = 0$, the stiffness the matrix becomes as the relation below after omitting the first and second rows and columns of the matrix in Equation (1):

$$[VL] = \frac{EI_2}{L} [A] \left\{ \frac{\delta}{L} \right\} \dots\dots\dots (13)$$

The non-dimensional axial load parameter making the stiffness matrix to vanish is obtained by trial and error with interpolation when equating Equation (14) to zero, as given in **Table (1)** for different tapering ratios.

$$S_1 + S_2 + 2SC - \pi^2 \rho_2 = 0 \dots\dots\dots (14)$$

3-5 Fifth Model

A beam column member is fixed at the top and pinned at the base, and loaded axially with side-sway. The elastic critical load is obtained by substituting the boundary condition which is $\theta_1 = 0$, the stiffness matrix becomes as the relation below after omitting the first row and column of matrix in Equation (1):

$$\begin{Bmatrix} \mathbf{M}_1 \\ \mathbf{VL} \end{Bmatrix} = \frac{\mathbf{EI}_2}{\mathbf{L}} \begin{bmatrix} \mathbf{S}_2 & \mathbf{S}_2 + \mathbf{SC} \\ \mathbf{S}_2 + \mathbf{SC} & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \frac{\delta}{\mathbf{L}} \end{Bmatrix} \dots\dots\dots (15)$$

The non-dimensional axial load parameter making the stiffness matrix to vanish is obtained by trial and error with interpolation when equating Equation (16) to zero, as given in **Table (1)** for different tapering ratios.

$$\mathbf{S}_2 \mathbf{A} - (\mathbf{S}_2 + \mathbf{SC})^2 = 0 \dots\dots\dots (16)$$

Table (1) Coefficients-C for elastic critical load for different support conditions

Tapering ratio, u	Hinged-hinged	Fixed-hinged	Fixed-free	Fixed-fixed with side-sway	Fixed-hinged with side-sway
1.00	1.000000	2.045752	0.250000	1.000000	0.250000
1.25	1.403221	2.686418	0.305275	1.408521	0.400397
1.50	1.861839	3.799353	0.359792	1.884961	0.588814
1.75	2.374610	4.835264	0.413758	2.430482	0.816179
2.00	2.940594	5.973663	0.467305	3.046091	1.083267
2.25	3.559051	7.212586	0.520523	3.732678	1.390733
2.50	4.229384	8.550445	0.573474	4.491039	1.739143
2.75	4.951100	9.985920	0.626206	5.321891	2.128983
3.00	5.723784	11.517898	0.678754	6.225885	2.560683
3.25	6.547083	13.145426	0.731146	7.203613	3.034618
3.50	7.420691	14.867678	0.783404	8.255618	3.551121
3.75	8.344343	16.683930	0.835545	9.382400	4.110491
4.00	9.317802	18.593542	0.887584	10.584418	4.712994
4.25	10.340863	20.595945	0.939533	11.862098	5.358870
4.50	11.413338	22.690629	0.991402	13.215834	6.048330
4.75	12.535063	24.877132	1.043199	14.645992	6.781574
5.00	13.705886	27.155038	1.094931	16.152913	7.558780
Buckled Shape of Columns					

From the above table, the elastic critical load of the prismatic and non-prismatic members can be calculated by using the suggested equation below:

$$Q_{cr} = C \frac{\pi^2 EI}{L^2} \dots\dots\dots (17)$$

This equation can be used as a general equation, where:

k : is the effective length factor (equals to 1.0 for prismatic members with pinned ends, and equal to 0.69916 for member fixed at one end and pinned at the other),

u : is taper ratio, $=b/a = d_1/d_2$

I : is the moment of inertia at any section of the prismatic member when $u=1$, and the moment of inertia at the smaller end of the non-prismatic member when $u > 1$, ($I = I_2$).

C : is the non-dimensional axial force parameter ρ .

4. Deflected Mode Shape of Pinned-Ends Column

The second order differential equation in terms of bending moments is given in Equation (18), which represents the differential equation of the tapered members' pinned-ends having rectangular or square cross section bent about the major axis, and subjected to axial load.

The solution of Equation (18) is given below in Equation (19), which is the deflected mode shape of the column in terms of Bessel functions ^[1, 5, 6].

$$EI_2 \left(\frac{x}{a} \right)^3 \frac{d^2 y}{dx^2} + Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \dots\dots\dots (18)$$

$$y(x) = \sqrt{x} \left[AJ_1 \left(\frac{2\omega}{\sqrt{x}} \right) + BY_1 \left(\frac{2\omega}{\sqrt{x}} \right) \right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \dots\dots\dots (19)$$

where:

$$A = -\frac{M_2 \cdot \sqrt{u} \cdot Y_1 \left(\frac{2\omega}{\sqrt{b}} \right) + M_1 \cdot Y_1 \left(\frac{2\omega}{\sqrt{a}} \right)}{Z \cdot Q \cdot \sqrt{b}} \text{ and } B = \frac{M_2 \cdot \sqrt{u} \cdot J_1 \left(\frac{2\omega}{\sqrt{b}} \right) + M_1 \cdot J_1 \left(\frac{2\omega}{\sqrt{a}} \right)}{Z \cdot Q \cdot \sqrt{b}}$$

5. Effective Length

The effective length is defined as the length of the equivalent pinned-end column, or the distance between points of inflection in the deflection curve. The effective length concept is one method for estimating the interaction effect of total frame on a column. This concept uses k -factors to equate the strength of frame compressed element of length L to an equivalent pin-ended member of length kL subject to axial load only. The critical load for columns with various support conditions can be related to the critical load of a pinned-ends column through the concept of an effective length.

The ratio, effective column length to the actual braced or unbraced length, may be greater or less than 1.0, therefore the effective length can be determined using Equation (20):

$$C_1 \frac{\pi^2 EI}{(kL)^2} = C \frac{\pi^2 EI}{L^2} \dots\dots\dots (20)$$

$$k = \sqrt{\frac{C_1}{C}} \dots\dots\dots (21)$$

where:

C : is obtained from Table (1) for any support conditions.

C_1 : is the Coefficients- C obtained from **Table (1)** for hinged-hinged support conditions.

The effective length values with respect to the different support conditions are tabulated at the tapering ratios from $u=1$ up to $u=5$ as given in **Table (2)**, this table can be used as a reference to obtain the effective length for any support case.

From **Table (2)**, the effective length of the non-prismatic columns for any tapering ratio is determined in five support conditions. The effective length is equal to 1.0 in any tapering ratio at hinged-hinged supports, while the effective length is less than 1.0 in fixed-hinged supports and it is very close to 1.0 for different tapering ratios in fixed-fixed with side-sway between supports. The effective lengths of the non-prismatic members in different tapering ratios are more than the effective lengths of prismatic shapes in case of fixed-free supports, but it is less than the effective length of prismatic shapes in fixed-hinged with side-sway between supports.

Table (2) Effective length for different support conditions

Tapering ratio, u	Fixed-hinged	Fixed-free	Fixed-fixed with side sway	Fixed-hinged with side sway
1.00	0.699155	2	1	2
1.25	0.699738	2.143964	0.998117	1.872051
1.50	0.700029	2.274811	0.993848	1.778206
1.75	0.700787	2.395648	0.988439	1.705703
2.00	0.701612	2.508519	0.982531	1.647592
2.25	0.70246	2.614852	0.976465	1.599725
2.50	0.703306	2.715699	0.970432	1.559448
2.75	0.704136	2.811851	0.964535	1.524982
3.00	0.704945	2.903925	0.958829	1.495078
3.25	0.705727	2.992415	0.953342	1.468831
3.50	0.706481	3.077721	0.948085	1.445571
3.75	0.707208	3.160175	0.94306	1.424785
4.00	0.707906	3.240052	0.93826	1.406074
4.25	0.708578	3.317587	0.933679	1.389127
4.50	0.709223	3.392981	0.929307	1.37369
4.75	0.709844	3.466408	0.925132	1.359559
5.00	0.710442	3.538019	0.921145	1.346566

6. Verification

Non-prismatic members are solved by using the finite elements method by dividing this non-prismatic members into different number of equivalent prismatic elements as shown in **Fig.(2)** under increasing axial load until the stiffness $K-K_g$ vanishes, where the flexural stiffness matrix is given in Equation (22), and geometric stiffness matrix is given in Equation (23) as defined below ^[7]:

$$[K] = EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \dots\dots\dots (22)$$

$$[K_g] = Q \begin{bmatrix} \frac{36}{30L} & \frac{1}{10} & -\frac{36}{30L} & \frac{1}{10} \\ \frac{1}{4L} & \frac{30}{10} & -\frac{1}{36} & -\frac{L}{30} \\ \frac{10}{-36} & -\frac{1}{30} & \frac{36}{10} & -\frac{1}{30} \\ \frac{30L}{1} & \frac{10}{-L} & \frac{30L}{-1} & \frac{10}{4L} \\ \frac{1}{10} & \frac{30}{30} & \frac{10}{10} & \frac{30}{30} \end{bmatrix} \dots\dots\dots (23)$$

The boundary conditions for the non-prismatic beam-column member have been applied to the system stiffness matrix $K-K_g$. The buckling load has been obtained for a specific discretization by increasing the compressive axial load incrementally until the stiffness $K-K_g$ vanishes.

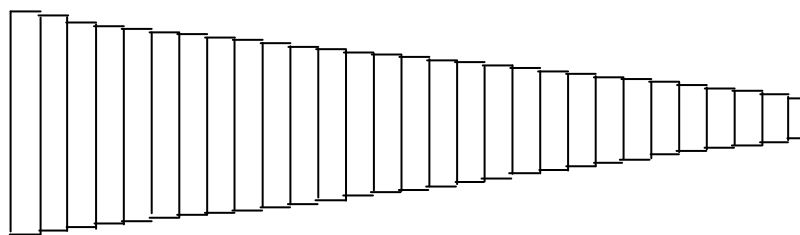


Figure (2) Non-prismatic member divided into large number of equivalent prismatic elements

7. Application

A beam-column member loaded axially is shown in **Fig.(3)**, it has 5m length, rectangular cross sectional shape with 0.025 m depth and 0.01 m widths at smaller end, 0.0375 m depth and 0.01 m widths at larger end, the modulus of elasticity is 200 GPa. The elastic critical load for this model is obtained as below:

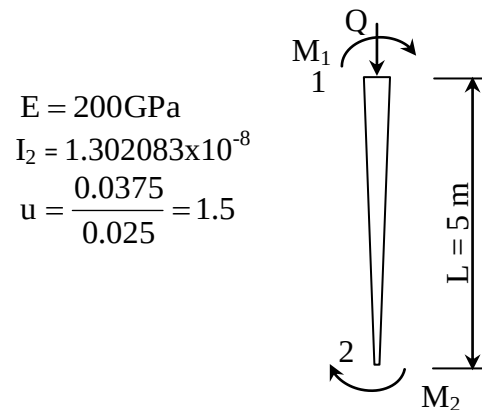


Figure (3) Properties of model

By substituting the five sets of boundary conditions in the stiffness matrix, the elastic critical loads are given in **Table (3)** with respect to the support conditions:

Table (3) Elastic critical load in different support conditions

Support Conditions	Coefficient C	Euler's Load kN	Q _{cr} , kN		Effective length
			Exact Method	Finite Elements Method	
Hinged-hinged	1.861839	1.028084	1.914127	1.856	1.000000
Fixed-hinged	3.799353	1.028084	3.906054	3.788	0.700029
Fixed-free	0.359792	1.028084	0.369896	0.358	2.274811
Fixed-fixed with side sway	1.884961	1.028084	1.937898	1.879	0.993848
Fixed-hinged with side sway	0.588814	1.028084	0.605350	0.587	1.778206

The coefficient C making the stiffness matrix to vanish is obtained from **Table (1)**, and the effective length is obtained by using **Table (2)**, where the critical load is equal to:

$$Q_{cr} = C \cdot Q_E$$

The Euler load for members with pinned ends is equal to:

$$Q_E = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 200,000,000 \times 1.302083 \times 10^{-8}}{5^2} = 1.028084 \text{ kN}$$

By using the finite element method the elastic critical load is calculated with different number of equivalent prismatic elements, the value of elastic critical load at the large numbers of elements is converged to that obtained from exact method.

8. Conclusion

The exact values of the elastic critical load of non-prismatic members with different support conditions are obtained by using the modified stability functions.

The elastic critical load increased with the increasing of tapering ratio in different support conditions. Also the ratio of the elastic critical load of the non-prismatic members with respect to that of prismatic members ranged between 438% in hinged-hinged support case and 3024% in fixed-hinged support with side-sway when the tapering ratio is equal to 5.

The effective length of the non-prismatic columns is obtained with respect to the hinged-hinged support that is equal to 1.0. The values of the effective length increased with the increasing of the tapering ratio in fixed-fixed and fixed-free supports while it decreased in fixed-fixed and fixed-hinged supports case with side-sway. The ratios of the effective length of the non-prismatic members to that of the prismatic members between 67.3% at fixed-fixed with side-sway and 176.9% at fixed-free support when tapering ratio is equal to 5.

The approximate elastic critical load by using the finite elements method is not more than 97% of the exact method when dividing the non-prismatic member into 250 elements.

This study enabled any engineer to estimate the elastic critical load and effective length for different support conditions with any value of tapering ratios by using **Table (1)** and **Table (2)**.

9. References

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Symbols

- M_1 & M_2 : are the clockwise moments in strut at end 1, and end 2 respectively
 V : is the end shear force
 θ_1 & θ_2 : are the clockwise rotations in strut at end 1, and end 2 respectively
 δ : is the clockwise displacement of end 2 perpendicular to strut
 $k = EI_2/L$: is the stiffness of the strut with respect to end 2
 E : is Young's modulus of column material
 I_1 & I_2 : are the moment of inertia at end 1, and end 2 respectively
 L : is the column length
 S_1 & S_2 : are the modified stiffness factor at end 1, and end 2 respectively
 SC : is the modified moment carry-over factor
 A : is the modified shear stiffness factor equal to $(S_1 + S_2 + 2SC - \pi^2 \rho_2)$
 u : Taper ratio, $=b/a = d_1/d_2$
 ρ_2 : is the non-dimensional axial force parameter with respect to the smaller end depth $(\frac{Q}{Q_E})$
 Q : is the axial force
 Q_E : is the Euler load for any member and equal to $\pi^2 EI / L^2$.