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Closed Form Solution for Partial Interaction of Composite Reinforced Concrete Beams

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Abstract

An analytical method for the elastic analysis of the interaction between a reinforced concrete beam and a steel plate bonded to it by flexible connectors is presented .The shear distributed force is formulated. Also, the ultimate strength, slip and deflection for different cases of loading are studied.

It is observed that large shear is concentrated at the edges of the steel plate. Comparing with some experimental results from references, this model gives close prediction.

الخلاصـــــة

يقدم البحث طريقة تحليلية للعتبات الخرسانية المركبة وفي هذه الدراسة سنتناول العتبات الخرسانية المسلحة والمزودة بصفائح فولاذية خارجية تربط إلى العتب بواسطة روابط مرنة تسمى " الروابط القصية " . حيث تم التوصل إلى معادلة تفاضلية بمتغير واحد هو القص العرضي الموزع، ومنه توصل إلى معادلات تحسب الانزلاق والانحراف في هذه المنشآت. كما تم أيجاد طريقة لحساب المقاومة القصوى لهذه العتبات والخواص الأخرى المختلفة.

البحث طبق على العديد من النتائج العملية المأخوذة من المصادر المثبتة في البحث حيث بينت النتائج تركز الاجهادات العمودية والقصية على حافة الصفيحة الفولاذية كما أن النتائج أعطت تقارب كبيراً من النتائج العملية التي أجريت من قبل باحثين.

1. Introduction

Normally, two categories of limit state are considered in design, the ultimate state and the serviceability states which are concerned with deflection and cracking under service load. The structures usually designed for the ultimate limit state and checked for the serviceability limit state. An ultimate state is reached when structure (or part of it) is collapsed .Collapse may rise from the rupture of one or more critical sections, due to excessive shear forces or bending moment. Shear failures are characterized by small deflection and lack of ductility, which makes it, one dangerous, due to the lack of warring. Therefore, in the design of concrete structures, it is generally desirable to ensure that flexure rather than shear govern the ultimate strength [1].

Previous work showed that the bonding of external thin plates to the tensile face of damaged or undamaged concrete beam can lead to a significant improvement in structural performance under both service and ultimate load condition and the failure of these beams is dominant by extensive flexural concrete cracking, yielding of internal reinforcement and external plate and crushing of concrete in the compression zone [2,3]. However, for thicker plates, severe shear and normal stresses concentration in the adhesive layer at the ends of plates may result in premature. This failure mode is indicated by ripping off the concrete cover together with the bonded plate, if epoxy resins have been used as bonding material.

Therefore an alternative method for attaching the external plate is suggested, i.e. using bolting technique. Little work has been encountered on concrete beam strengthened by mechanically attached steel plate.

2. Ultimate Strength

The failure load of the composite beams is predicted assuming bending failure and that a sufficient number of shear connectors were provided at the concrete-steel plate interface, such that the flexural strength is not reduced due to deformation of shear connectors. Two different approaches are represented to estimate the ultimate strength of such composite beams , which are based on either the American Standard (ACI-318-89) [4] or British Standard (BS 8110) [5].

Considerations for both approaches, the partial safety factors are omitted from the calculation. The ACI code-based approach is characterized by the usage of Whitney's stress block with maximum compressive concrete strain of (0.003) as shown in **Fig.(1)**.

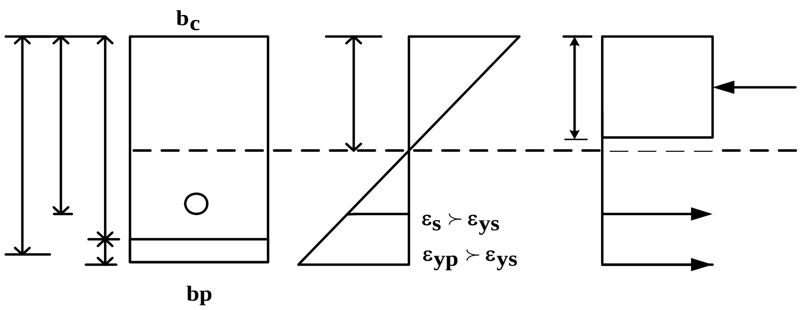

Figure (1) Strain and stress distribution of failure (ACI Building Code)

The tension components of the moment couple are contributed by the reinforcing bars (T_s) and steel plate (T_p) and are defined as:

$$T_s = As . fy_s , T_p = A_p .fy_p \quad \text{.......... (1)}$$

where: *(A_s, fy_s) and (A_p, fy_p) denote the sectional area and yielding stress of the conventional steel reinforcement and steel plate, respectively.*

The depth of the concrete rectangular stress block (a_c) can be found after satisfying the equilibrium of the horizontal forces:

$$a_c = \frac{As . fy_s + A_p . fy_p}{0.85 f'c , b_c} \quad \text{.......... (2)}$$

where, *(f_c') and (b_c) are concrete compressive strength of cylinder and width of the concrete section, respectively.*

Then the ultimate moment of resistance (M_u), is given by:

$$Mu = A_s . fy_s .(d_s - a_c /2) + A_p . fy_p .(d_p - a_c / 2) \quad \text{.......... (3)}$$

where, *d_s and d_p are effective depths of the conventional steel reinforcement and steel plate, respectively.*

Since steel plate are used externally to strengthen an existing damaged reinforced concrete section with a predesigned steel ratio (A_s, b_c, d_s) .There should be a limit introduced or the sectional area of the steel plate to be added in order to ensure a ductile tension failure (under-reinforced section) for the composite section.

The A_p can be determined by using equations (2) and (3) as:

$$A_p = \frac{- K2 + \sqrt{K2^2 - 4. K1.K3}}{2.K1} \quad \text{.......... (4)}$$

$$K1 = \frac{fy_p^{\,2}}{1.7 f_c' .b_c} ; \; K2 = \frac{T_s \;\; fy_p}{0.85 f'c \;\; .b_c} - fy_p d_p , \text{and } K3 = \frac{T_s^{\,2} \;\; f_{yp}}{1.7 f' c.b_c} - d_s T_s + M_u$$

The value (A_p) obtained from equation (4) above should be limited to the balanced steel plate area (A_{pb}) which can be defined as the plate area at the balanced load conditions and also the maximum plate area to insure a ductile tension failure. Balanced conditions are defined such that internal steel reaches concurrently with the extreme concrete fiber strain. For any given section (A_{pb}) can be found, and will be based on the strain distribution shown in **Fig.(2)**.

$$Y_b = \frac{0.003}{0.003 + \varepsilon_{ys}} d_s \quad \text{.......} (5)$$

while, $a_{cb} = y_b\,\beta 1$ then:

$$A_{pb} = \frac{0.85 f'_c b_c \beta 1 - A_s f_{ys}}{f_{yp}} \quad \text{.......} (6)$$

where:

$\beta 1 = 0.85$ *For* $f'_c \le 30\ N/mm^2$

$\beta 1 = 0.85 - 0.008\,(f'_c - 30)$ *For* $f'_c > 30\ N/mm^2$

also, β1 not less than (0.65).

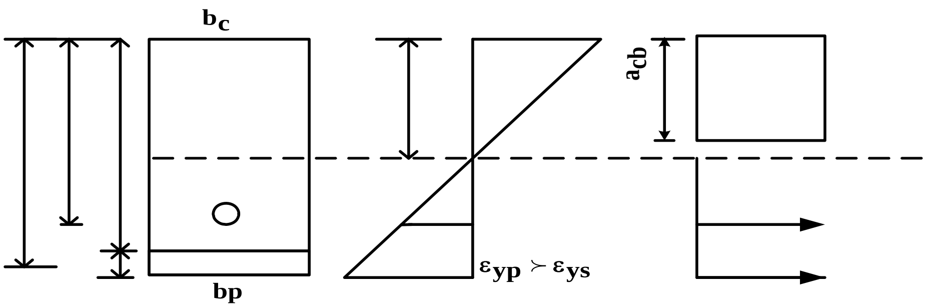

Figure (2) Balanced load condition

The second approach is based on the British Standard consideration with the stress block of ($0.67f_{cu}$) by (0.9y), while failure strain for concrete is taken as (0.0035), see **Fig.(3)**. Therefore satisfy horizontal equilibrium, neutral axis depth (y) can be calculated and the ultimate moment of resistance (Mu) is given by:

$$Mu = A_s.fy_s\,(ds - 0.45y) + A_p.fy_p\,.(d_p - 0.45y) \quad \text{.......} (7)$$

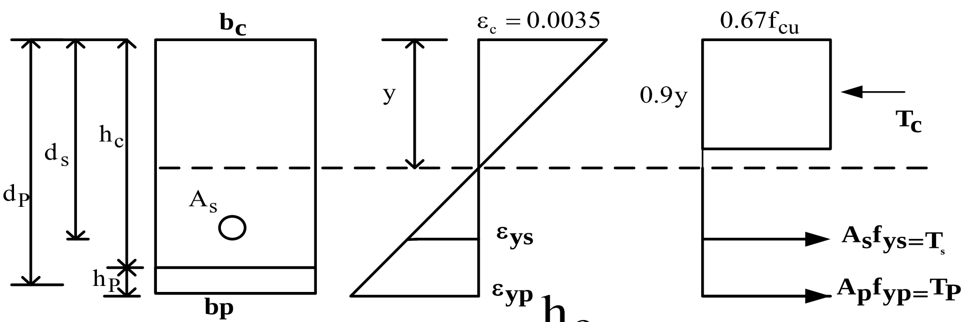

Figure (3) Strain and stress distribution of failure (BS 8110 Block Stress)

y

hc

ds

dP

As

hP

3. Properties of Elastic Transformation Section

The stress and strain distribution a cross the depth of the platted beam for elastic uncracked and cracked section are shown in **Fig.(4)** based on usual assumptions for elastic range.

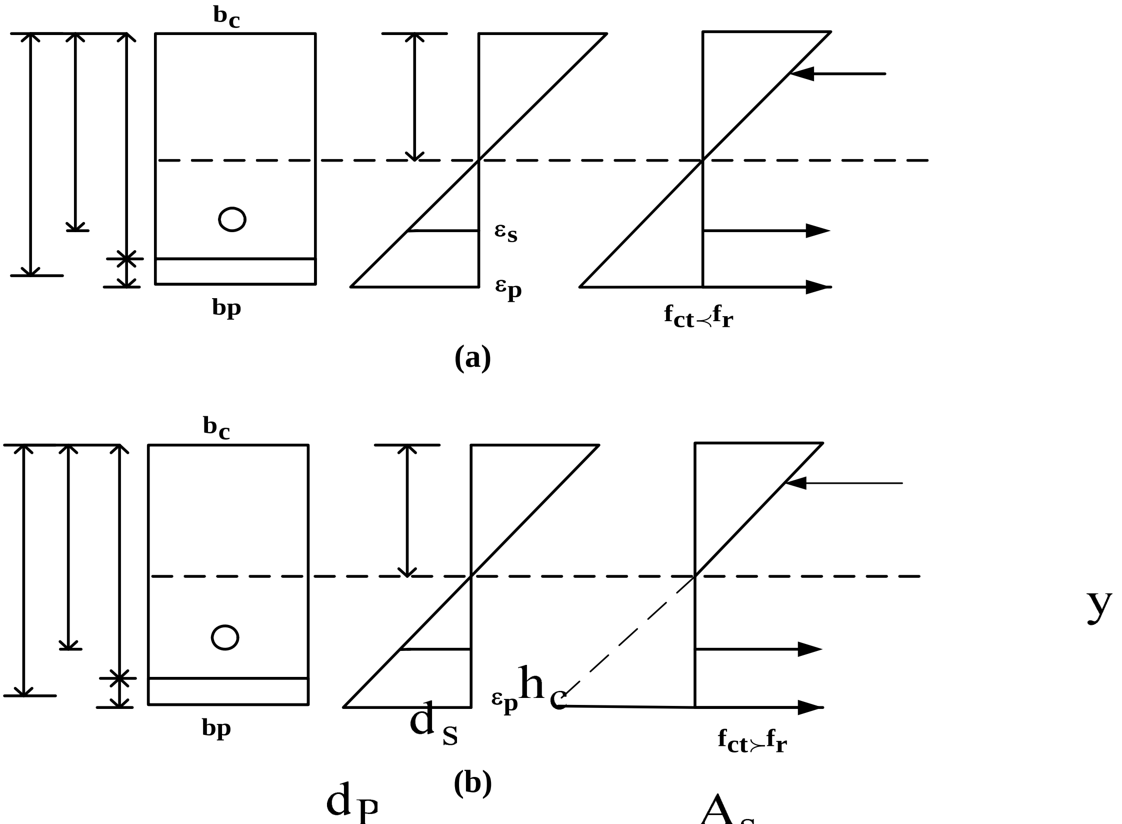

Figure (4) Stress and distribution a cross the depth of beam for elastic analysis
a) Uncracked section b) Cracked section

For elastic uncracked section, assuming that the concrete can sustain tension and neglecting the conventional tensile reinforcement, the depth of the neutral axis (y) from the top of sectional area can be found by taking moment of area of the transformed section as:

$$y = \frac{\frac{b_c.h_c^{\,2}}{2} + n_p A_p.d_p}{b_c.h_c + n_p A_p} \quad \text{.. (8)}$$

The second moment of area for the equivalent steel is given by:

$$I = \frac{b_c h_c^{\,3}}{12} + b_c h_c\left(\frac{h_c}{2} - y\right)^2 + n_p A_p (d_p - y)^2 \quad \text{...................................... (9)}$$

The moment of inertia of the external steel plate about its own axis is neglected.

The section properties of the concrete beam and steel plate alone are given by:

$$E_{c_1} I_{c_1} = E_c \frac{b_c\, h_c^3}{12}, \; E_p I_p = E_p \frac{b_p\, h_p^3}{12}$$

However, for elastic cracked section, the depth of the neutral axis (y) can be defined by satisfying the horizontal force equilibrium as in **Fig.(4)**.

$$\mathbf{F_c = T_s + T_p} \quad \text{...} \quad \mathbf{(10)}$$

where, $F_c = 0.50 .\, f'_c .\, b_c .y$; $T_s = A_s .f_s$ and $T_p = A_p .f_p$

The value of f_s and f_p can be determined from linear stress distribution, then substituting these values in equation (10), result in:

$$\mathbf{\frac{b_c y_2}{2n_s} = y\left[\frac{n_p}{n_s} A_p + A_s\right] - \left[\frac{n_p}{n_s} A_p d_p + A_s d_s\right] = 0} \quad \text{.....................................} \quad \mathbf{(11)}$$

$$\mathbf{I = \frac{b_c\, y^3}{3} + n_s A_s (d_s - y)^2 + n_p A_p (d_p - y)^2} \quad \text{.......................................} \quad \mathbf{(12)}$$

Once (y) is known, the moment of inertia for elastic cracked section (I) can be calculated using the usual expression,

The section properties of the reinforced concrete beam and steel plate alone are given by:

$$E_{c_1} I_{c_1} = E_c \frac{b_c\, \bar{y}^3}{3} + E_s A_s (d_s - \bar{y})^2, \; E_p I_p = E_p \frac{b_p\, h_p^3}{12}$$

where: y is determined from equation (11) with $h_p = d_p = 0$.

4. Shear Distributed Load

The equilibrium of a differential section of the reinforced concrete beam with reinforcing steel plate is shown in **Fig.(5)**. By substituting a transformed concrete section substitute of reinforced concrete beam which have moment of inertia, modulus of elasticity and area denote by Ic1, Ec1 and Ac1 respectively. Shear distributed forced (q) as well as distributed peeling forced (T) are acting between the concrete beam (i.e. transformed section) and steel plate.

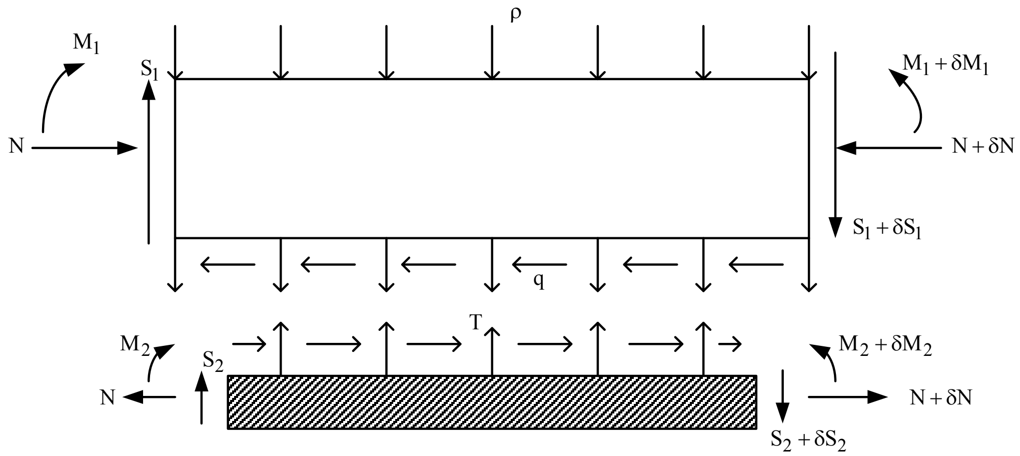

Figure (5) Composite finite element of the beam

The longitudinal equilibrium of either the concrete beam or steel plate gives:

$$\mathbf{N_{,x} = -q} \quad \text{……………………………………………………………….. (13)}$$

The equilibrium of vertical forced implies:

$$\mathbf{S_{1,x} = T + p}\ ,\quad \mathbf{S_{2,x} = -\,T} \quad \text{For upper and lower element ……..…..………………….. (14)}$$

The equilibrium of moment of concrete of and steel plate about their centroids

$$\mathbf{M_{1,x} = q.h_1\ \ S_1} \qquad \text{For upper element ………………………………….. (15)}$$

$$\mathbf{M_{2,x} = q.h_2\ \ S_2} \qquad \text{For lower element ………….............................. (16)}$$

Also the equilibrium of concrete finite element satisfies compatibility equation:

$$\mathbf{q = \frac{U_{cs}\,.K}{S}} \quad \text{……..……………………………………………………. (17)}$$

where: *Ucs is the slip at the interface between the concrete element and steel plate element, and K is shear stiffness of the connectors and S is the spacing between connectors.*

Derivative equation (17) once with respect to x gives:

$$\mathbf{q_{\ ,x} = U_{cs\ ,x}\,.\,K_s}$$

……………………………………………………………….. (18)

where, *Ks = k/s and Ucs, x is defined as the slip strain at the interface between the concrete and steel plate and defined:*

$$U_{cs,x} = \varepsilon_1 - \varepsilon_2 \quad \text{(19)}$$

where: ε_1 *represent the strain at the bottom of concrete beam and* ε_2 *is the strain at the top of steel plate.*

Assuming equal curvatures for the two elements given

$$w_{c1,xx} = w_{p,xx} = w_{,xx} \quad \text{(20)}$$

From elastic beam theory:

$$w_{,xx} = w_{c1,xx} = \frac{M_1}{E_{c_1} I_{c_1}} \quad \text{(21)}$$

$$w_{,xx} = w_{p,xx} = \frac{M_2}{E_p I_p} \quad \text{(22)}$$

Now, ε_1 and ε_2 can be defined as below:

$$\varepsilon_1 = h_1 \frac{M_1}{E_{c_1} I_{c_1}} - \frac{N}{E_{c_1} A_{c_1}} \quad \text{(23)}$$

$$\varepsilon_2 = -h_2 \frac{M_2}{E_p I_p} - \frac{N}{E_p A_p} \quad \text{(24)}$$

By using equations (18), (19), (23), and (24) yields:

$$\frac{q_{,x}}{K_s} = h_1 \frac{M_1}{E_{c_1} I_{c_1}} + h_2 \frac{M_2}{E_p I_p} - N\left[\frac{h_1}{E_{c_1} A_{c_1}} + \frac{h_2}{E_p A_p}\right] \quad \text{(25)}$$

When derivative equation (25) once with respect to x and substituting the values of $N_{,x}$, $M_{1,x}$, and $M_{2,x}$ from above equations, get:

$$q_{,xx} = qK_s\left[\frac{h_1^{\,2}}{E_{c_1} I_{c_1}} + \frac{1}{E_{c_1} A_{c_1}} + \frac{h_2^{\,2}}{E_p I_p} + \frac{1}{E_p A_p}\right] - \frac{h_1 S_1 K_s}{E_{c_1} I_{c_1}} - \frac{h_2 S_2 K_s}{E_p I_p} \quad \text{(26)}$$

It is assumed that the shear due to external load, Sh, is carried by concrete only, so that

$$\mathbf{S_1 = Sh,\ S_2 = 0}$$

……………………………………………………………… **(27)**

Let: $\lambda^2 = Dn.K_s$, where, $Dn = \frac{h_1{}^2}{E_{c_1} I_{c_1}} + \frac{1}{E_{c_1} A_{c_1}} + \frac{h_2{}^2}{E_p I_p} + \frac{1}{E_p A_p}$

By using equations (27), the effect of the peeling distributed forced on the deflection of the concrete beam is ignored, equation (26) becomes:

$$\mathbf{q_{,xx} - \lambda^2 K_s q + \frac{h_1 K_s}{E_{c_1} I_{c_1}} Sh = 0} \quad \text{………………………………………… (28)}$$

5. Solution for Uniformly Distributed Load

For a simply supported beam of span L strengthed mechanically with external steel plate, its length is $\mathbf{l_s}$, and subjected to a uniformly distributed load p see **Fig.(6)**. The distance x is measured from the beginning of the steel plate. The shear force Sh, due to external load at distance x from the left end of the steel plate is: (down direction is positive)

$$\mathbf{Sh = p.(a + x - L/2)} \quad \text{………………………………………………… (29)}$$

where, *(a) is the distance from the support to the nearer beginning of steel plate.*

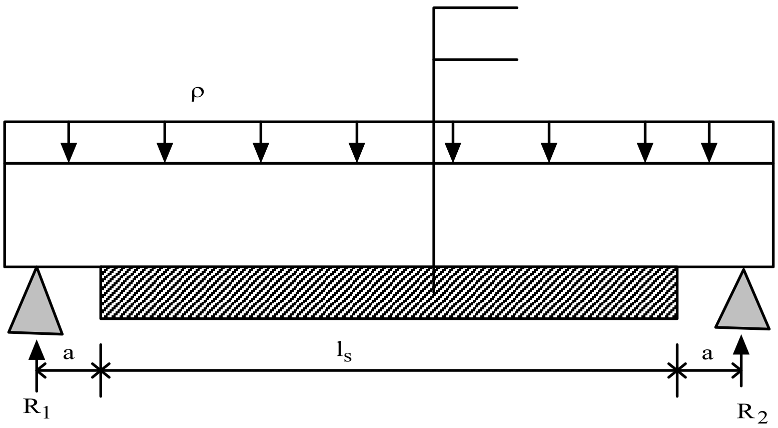

Figure (6) Strengthened concrete beam with uniformly distributed load

Substituting value of Sh from equation (29) in to equation (28) the general solution of the differential equation (28) can be written:

$$\mathbf{q = A.\cosh \lambda x + B.\sinh \lambda x + \frac{h_1 \rho}{E_{c_1} I_{c_1} .Dn}(x + a - L/2)} \quad \text{……………………… (30)}$$

where: *A and B are the constants of integration, and their values are obtained by using the boundary conditions.*

In case of reinforced concrete beam with external steel plate, $1/\lambda$ indicates the length in which the large shear stress at the edges drops to its average value.

5-1 Boundary Conditions

1. At $x = 0$, $q_{,x} = \dfrac{M_1 K_s . h_1}{E_{c_1} I_{c_1}}$ where, $M_1 = (\dfrac{p.a}{2}).(L - a)$

2. At $x = l_s/2$, $q = 0$

After applying the boundary conditions the constants A and B can be determined, substituting these values in equation (30), yield:

$$\mathbf{q = \frac{h_1 \rho}{E_{c_1} I_{c_1} .Dn}\left[\left(\frac{K_s.a.(L-a)}{2\lambda} - \frac{1}{\lambda Dn}\right)\left(\sinh\lambda x - \tanh\frac{\lambda l_s}{2}\cosh\lambda x\right) + \frac{a + x - \frac{L}{2}}{Dn}\right] \quad ..... (31)}$$

6. Solution for Eccentric Point Load

The same beam in prior section, may now subjected to a point load P applied at distances l_1 and l_2 from the ends of the steel plate as shown in **Fig.(7)** .The shear force to the left and right of the load are given by

$$Sh = -p\left(\frac{l_2 + a}{ls + 2.a}\right) \quad \text{Left of point load,}$$

$$\mathrm{Sh} = -\mathrm{p}\left(\frac{\mathrm{l}_1 + \mathrm{a}}{\mathrm{ls} + 2.\mathrm{a}}\right) \quad \text{Right of point load}$$

Solving equation (29), for the two parts of composite platted beam given:

$$\mathbf{q_L = A1.\cosh\lambda x + B1.\sinh\lambda x - \frac{(l_2 + a)h_1 P}{(l_s + 2a)E_{c_1} .I_{c_1} Dn}} \quad \text{.............................. (32)}$$

$$\mathbf{q_R = A2.\cosh\lambda(x-1) + B2.\sinh\lambda(x-1) - \frac{(l_2 + a)h_1 P}{(l_s + 2a)E_{c_1} .I_{c_1} Dn}} \quad \text{................. (33)}$$

where, A1,A2, B1 and B2 are the constants of integration, q_L and q_L and q_R denoted to the distribution shear forces in the left and right of the load respectively.

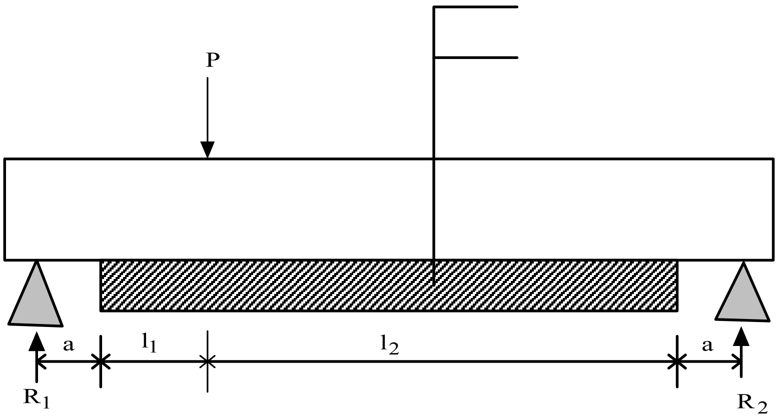

Figure (7) Strength concrete beam with point load

6-1 Boundary Conditions

1. $x = l_1$ at $q_L = q_R$ and $q_{L,x} = q_{R,x}$

2. $q_{L,x} = \dfrac{K_s.\ h_l.M_1}{E_{c_1}.I_{c_1}}$ at x= 0

3. $q_{R,x} = \dfrac{K_s.\ h_l.M_1}{E_{c_1}.I_{c_1}}$ at x=l_s where, $M_1 = \dfrac{P(l_2 + a)}{(l_s + 2.a)}a$

After apply the boundary conditions, the values of constants of integration can be found and substituting in equations (32) and (33), yield:

$$\mathbf{q_L = R_0 \operatorname{csch}\lambda l_s((l_1 + a) - (l_2 + a)\cosh\lambda l_s)\cosh\lambda x}$$
$$\mathbf{+ R_0(l_2 + a)\sinh\lambda s + R_1\left[\left(\frac{\sinh \lambda l_2}{\sinh\lambda L_s}\right)\cosh\lambda x - \left(\frac{l_2 + a}{l_s + 2a}\right)\right]} \quad \text{.....................} \mathbf{(34)}$$

$$\mathbf{q_R = R_0 \operatorname{csch}\lambda l_s[((l_1 + a)\cosh\lambda l_1 - (l_2 + a)\cosh\lambda l_s)\cosh\lambda(x - 1)}$$
$$\mathbf{+ (l_1 + a)\cosh\lambda l_1 - (l_2 + a)\cosh\lambda l_2 .((l_1 + a)\sinh\lambda l_1} \quad \text{.....} \mathbf{(35)}$$
$$\mathbf{+ (l_2 + a)\sinh\lambda l_2)\sinh(x - l_1)] + R_1\left[\left(\frac{l_1 + a}{l_s + 2a}\right) - \left(\frac{\sinh\lambda l_1}{\sinh\lambda l_s}\right)\cosh(l_s - x)\right]}$$

where, $R_0 = \dfrac{K_s.a.h_1.P}{E_{c_1}\ .I_{c_1}.\lambda.(l_s + 2a)}$, and $R_1 = \dfrac{h_1.P}{E_{c_1}I_{c_1}Dn.(l_s + 2a)}$

7. Solution for Central Point Load

For the details of platted beam in prior section but loaded with central point load (P). then the distances (l_1) and (l_2) which are equal to ($l_s/2$), see **Fig.(7)**. Also it useful to use one side of beam due to its symmetry. The shear force due to external load is:

$Sh = -\frac{p}{2}$ left side, $Sh = \frac{p}{2}$ right side Solving equation 36 for part AB:

$$\mathbf{q = A3.\cosh\lambda x + B3\sinh\lambda x - R_1 = \frac{h_1.P}{2E_{c_1}I_{c_1}Dn}} \quad \text{.................................. (36)}$$

where, A3 and B3 are constants of integration, for part BC similar result can be obtained due to symmetry.

7-1 Boundary Conditions

1. $q_{,x} = \frac{K_S.h_1.M_1}{E_{C_1}I_{C_1}}$ at x= 0 where, $M_1 = \frac{P.a}{2}$

2. $q_{,x} = 0$ at x = $l_S/2$

After applying the boundary conditions, the constants A3 and B3 can be found and substituting the values of constants in equation (36) and simplified the terms, it can be reached to the following form:

$$\mathbf{q_L = Q_0(\text{sech}\frac{\lambda.l_s}{2}\cosh\lambda x - 1) + B3.(\sinh\lambda x - \tanh(\lambda l_s/2)\cosh\lambda x)} \quad \text{............ (37)}$$

where: $\mathbf{Q_0 = \frac{p.h_1}{2E_c.I_e.D}}$

8. Prediction of Slip

Slip can be defined as the longitudinal differential displacement at the interface between the concrete and steel plate. Because of flexibility of shear connectors and the crushing of the surrounding concrete, horizontal slip (Ucs) at the interface cannot be completely prevented. Also, slip is related to the shear flow as indicated in equation (17), and then slip can be found for different cases as:

8-1 Platted Beam with Uniformly Distribution Load

$$\mathbf{Ucs = \frac{h_1P}{E_cI_c}\left[\left(\frac{a.(L-a)}{2\lambda} - \frac{1}{\lambda^3}\right).\left(\sinh\lambda x - \tanh\frac{\lambda.l_s}{2}\cosh\lambda x\right) + \frac{x - l_s/2}{\lambda^2}\right]} \quad \text{....... (38)}$$

8-2 Platted Beam with Eccentric Point Load

For left side:

$$U_{csL} = \frac{R_0}{K_s} \operatorname{csch}\lambda l_s((l_1 + a) - (l_2 + a)\cosh\lambda l_s)\cosh\lambda x$$
$$+ \frac{R_0}{K_s}(l_2 + a)\sinh\lambda x + \frac{R_1}{K_s}\left(\left(\frac{\sinh\lambda l_2}{\sinh\lambda l_s}\right)\cosh\lambda x - \left(\frac{l_2 + a}{l_s + 2a}\right)\right) \quad \ldots\ldots\ldots\ldots (39)$$

For right side:

$$Ucs_R = \frac{R_0}{Ks} \operatorname{csch}\lambda l_s[((l_1 + a)\cosh\lambda l_1 - (l_2 + a)\cosh\lambda l_2)\cosh\lambda(x - l_1) +$$
$$(l_1 + a)\cosh\lambda l_1 - (l_2 + a)\cosh\lambda l_2((l_1 + a)\sinh\lambda l_1 \quad \ldots. (40)$$
$$+ (l_2 + a)\sinh\lambda l_2)\sinh(x - l_1) + \frac{R_1}{Ks}\left(\left(\frac{l_1 + a}{l_s + 2a}\right) - \left(\frac{\sinh\lambda l_1}{\sinh\lambda l_s}\right)\cosh(l_s - x)\right)$$

8-3 For Platted Beam with Central Point Load

$$Ucs = \frac{Q_0}{K_s}\left(\operatorname{sech}\frac{\lambda l_s}{2}\cosh\lambda x - 1\right) + \frac{B3}{Ks}\left(\sinh\lambda x - \tanh\frac{\lambda l_s}{2}\cosh\lambda x\right) \quad \ldots\ldots\ldots.. (41)$$

9. Prediction of Deflection

One of the important parameters at the service live of the structures is the value of deflection, which should be limited to satisfy an acceptable behavior. Therefore, prediction of deflection is an important step in design and checking, structural members. Noting that the summation of equations (15) and (16) gives,

$$M_{1,x} + M_{2,x} = d.q - Sh \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (42)$$

where, $d = h_1 + h_2$

Furthermore, from equations (21) and (22) the following equation can be obtained:

$$w_{,xxxx} = \frac{1}{H_0}(d.q_{,x} - Sh_{,x})\,m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

By using equation (43), it can reach to the following equation that represents the deflection at any position,

$$w = \left[\frac{d}{\lambda^3}(A.\sinh\lambda x + B\cosh\lambda x) + \rho\frac{x^4}{24}\left(\frac{d.h_1}{E_{c_1} I_{c_1} Dn} - 1\right)\right] \quad \text{.......................... (44)}$$
$$+ C1\frac{x^3}{6} + C2\frac{x^2}{2} + C3x + C4$$

9-1 Boundary Conditions

Since, the platted beam consists of two parts, the first part is reinforced concrete only, extend from supported to beginning and end of plate, and the second is platted part (reinforced concrete with steel plate). Then the first part always satisfying the simple bending theory, as given below:

$$w_{0,xx} = \frac{M_0}{E_{c_1} I_{c_1}} \quad \text{.. (45)}$$

where, M_0, w_0 *is the moment and curvature at any part of unplated beam,* M_0 *is defined as:*

$$M_0 = Ra.X_0 - \frac{\rho.X_0{}^2}{2}$$

where, *Ra is the reaction force at support,* X_0 *is available distance from the support to the free end of plate and (value of* X_0 *zero to value of (a)).*

Then the slope and deflection at any point of unplated part can be obtained from equation 45 as:

$$w_{0,x} = \frac{1}{E_{c_1} I_{c_1}}\left(Ra\frac{X_0{}^2}{2} - \frac{P.X_0{}^3}{6}\right) + C5 \quad \text{.. (46)}$$

$$w_0 = \frac{1}{E_{c_1} I_{c_1}}\left(Ra\frac{X_0{}^3}{6} - \frac{P.X_0{}^4}{24}\right) + C5X_0 + C6 \quad \text{...................................... (47)}$$

where, *C5 and C6 are constants of integration, then, there are five unknown constants and the following are five boundary conditions,*

$1-\ At\ \ x = l_s/2\ \ ;\qquad w_{,xx} = 0\ \ and\ \ w_{,x} = 0$

$2-\ At\ \ x = 0\ \ ;\qquad w_{,xx} = w_{0,xx},\quad w_{,x} = w_{0,x}\ \ and\ \ w = w_0$

After apply the boundary conditions above, the values of constants C1 to C5 are obtained.

10. Solution for Point Load

In this case, when point load at any position of platted beam, see figure 7, the $Sh_{,x}$ will equal zero and $q_{,x}$ is defined by either equations 32 or 33 the deferential equation for deflection is therefore;

$$\mathbf{w}_{,xxxx} = \frac{\mathbf{d.q}_{,x}}{\mathbf{H}_0} \quad \text{..........} \quad \mathbf{(48)}$$

Equation (48) is applied separately to the left and right of the point load and integrating four times provides two equations for w with eight constants of integration in addition to two constants for left and right unplated part.

10-1 Boundary Conditions

The following boundary conditions will help to obtained the constants of this case,

1. $At \quad x=0, \qquad w_{L,xx} = w_{0L,xx}\ , \quad w_{L,x} = w_{0L,x} \quad and \quad w_L = w_R$
2. $At \quad x=l_S, \qquad w_{R,xx} = w_{0R,xx}\ , \quad w_{R,x} = w_{0R,x} \quad and \quad w_R = w_{0R}$
3. $At \quad x=l_1, \qquad w_{L,xx} = w_{R,xx} \quad , \quad w_{L,x} = w_{R,x} \quad , w_{R,xx} = \frac{P}{E_c I_0} \ and \quad w_L = w_R$

11. Solution for Central Point Load

The deferential equation for the deflection that will be used here is the same as that used in prior section. The following equations that can be used for one part and reverse the results in the other part:

$$\mathbf{w} = \frac{\mathbf{d}}{\lambda^3.\mathbf{H}_0}\left[\mathbf{A3.sinh}\lambda\mathbf{x} + \mathbf{B3.cosh}\lambda\mathbf{x}\right] + \mathbf{C1}\frac{\mathbf{x}^3}{\mathbf{6}} + \mathbf{C2}\frac{\mathbf{x}^2}{\mathbf{2}} + \mathbf{C3x} + \mathbf{C4} \quad \text{..........} \quad \mathbf{(49)}$$

where, *A3 and B3 defined as above and C1 to C4 are constants of integration in addition to that obtained from the bending theory for unplated part.*

All constants of integration can be obtained by applying the following boundary conditions.

11-1 Boundary Conditions

1. $At \quad x = l_S/2, \qquad w_{,xxx} = \frac{P}{2E_{c_1} I_{c_1}} \quad and \quad w_{,x} = 0$
2. $At \quad x=0, \qquad w_{,xx} = w_{0,xx} \quad , w_{,x} = w_{0,x} \quad and \qquad w = w_0$

12. Length of Steel Plate

It can be concluded that the length of steel plate may be limited; i.e .the steel plate will not be extended to supports .The length should satisfy a certain limit in order to prevent premature failure. This limit is based on considering the reinforced concrete section alone to resist the applied moment .This case will exist at the plate end , which is distanced from the support by (a). For the case of a simple span with a central point load of (Pu), the maximum cut-off length must satisfy the following limit:

$$\mathbf{a_{max} \leq \frac{2.M_{ui}}{P_u}} \quad \text{...} \quad \mathbf{(51)}$$

where, M_{ui} *is the ultimate moment of resistance of the concrete section with bar reinforcement alone*

13. Results

From the previous work, it can be seen that the maximum shear stress is concentrated on the edges of steel plate and increased as the distance of the strengthening steel plate increased. Also, the increasing in cross section area of the steel plate led to decrease the central deflection and increase in ultimate strength.

A comparison between the calculated and the measured values obtained from literature. [3,4] are made. The ultimate load, maximum slip and defection are presented in **Tables (1)** and **(2)**, show a close agreement for all value that test are observed.

Table (1) A comparison between the values of shear and slip obtained from the present investigation with the experimental values of [Ref.6]

Beam	Axial shear (kN)	Theoretical Slip (mm)	Experimental Slip (mm)	Experimental/Theoretical
B3	50.00	0.25	0.31	1.24
B4	83.08	0.27	0.23	0.85
B5	70.00	0.49	0.24	0.49
B6	54.00	0.27	0.3	1.11
B7	151.10	0.37	0.41	1.10
B8	197.20	0.35	0.52	1.48
B9	72.26	0.28	0.37	1.30
B10	123.89	0.35	0.44	1.25
B11	148.38	0.23	0.27	1.20
B12	48.00	0.24	0.33	1.37

Table (2) Comparison between experimental [Ref.6,7] and theoretical result

Beam	Experimental Values		Present Model Values			Experimental /Present Model		
	Ult Load (kN)	Service load	Ultimate load (kN)		Service load	Ultimate load		Service load
		Deflec. (mm)	ACI	BS	Deflec. (mm)	ACI	BS	Central deflection
B3	88	2.40	78.60	76.30	3.20	1.10	1.15	0.75
B4	110	2.15	95.60	92.00	2.50	1.14	1.19	0.86
B5	111	2.10	96.00	92.70	2.26	1.15	1.20	0.93
B6	97	2.20	83.50	81.00	1.90	1.16	1.20	1.15
B7	110	2.45	95.20	93.00	4.28	1.15	1.18	0.57
B8	92	3.00	94.80	93.40	4.71	0.96	0.99	0.64
B9	89	2.61	78.20	75.80	3.162	1.13	1.17	0.83
B10	82	2.72	78.20	75.80	3.82	1.05	1.08	0.71
B11	142	2.50	134.4	129.2	2.86	1.06	1.10	0.87
B12	85	2.50	77.70	76.10	2.25	1.1	1.13	1.11
R1	19.5	9.50	15.30	14.70	8.20	1.27	1.33	1.16
C1	33.00	8.60	27.00	25.80	8.69	1.22	1.28	0.99
C2	35.30	9.40	33.50	32.00	10.67	1.05	1.10	0.88
C3	32.20	8.20	27.00	25.80	8.51	1.19	1.25	0.96
C4	46.80	7.70	34.30	32.80	10.46	1.18	1.24	0.74
D1	31.80	11.00	27.30	26.30	8.60	1.16	1.21	1.37
D2	33.50	11.00	33.00	31.50	10.42	1.02	1.06	1.06
D3	32.30	8.40	28.80	27.60	8.43	1.12	1.16	0.99
D4	37.00	11.20	35.50	31.80	10.98	1.10	1.16	1.02

14. References

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Summary and Concluding Remarks

Theoretical model was presented to predict the failure load, slip and deflection values for reinforced concrete beams with external steel plates: Failure load has been predicted using two approaches based on the ACI recommendations and BSI recommendations. Limitation has been introduced on the sectional area of steel plate provided in the tension face of the beam, which is based on derived balanced area in order to prevent compression failure of the composite section.

With respect to slip predictions, the current model calculates slip value along the composite length of the beam, after defining the shear values at the concrete-steel plate interface for different cases of loading.

Also, the presented model covers the deflection prediction by reaching the differential equation solved by integration after substituting the value of axial shear strain. The boundary conditions are fixed for each case.