Model Reduction of Continuous Systems by Modified Routh Array and Pade Approximation Technique

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Abstract

Model reduction for reducing a high-order transfer function to one of low order is based on combination of Pade approximation to calculate the numerator coefficients and modified Routh array to calculate the denominator coefficients of reduced order models.

The purpose of this paper is to show the combination of Pade and Routh methods satisfied both the transient and steady state parts of time response for reduced models. Many steps by using computer algorithm for evaluating the numerator and denominator coefficients of reduced model are developed.

It is shown that the time response of the proposed model is more closely following the original system response than those of the other models. Several examples are used to illustrate the procedure and to test the viability of such an approach to model reduction.

الخلاصة:

يتناول البحث اختزال الانظمة المستمرة ذو الدالة التحويلية العالية الى دالة ذو درجة اقل اعتمادا على تقريب بادا وذلك لحساب معاملات البسط و باستخدام طريقة راوث المحورة لحساب معاملات المقام للدالة المبسطة.

تم استخدام خوارزمية لهذا الغرض. ان الهدف الرئيسي من هذا البحث هو اظهار لو تم جمع طريقة راوث المحورة مع تقريب بادا سوف يعطي دالة مختزلة ذو مواصفات مطابقة للدالة الاصلية عند الاستجابة العابرة و الحالة الثابتة لكلا الدالتين(الاصلية والمختزلة). تم اظهار فعالية هذة الطريقة من خلال الاستعانة بعدة امثلة.

1-Introduction

It is often desirable and sometimes necessary, for analysis and design purpose to reduce the order of the transfer function of a complex system. It is necessary of model reduction technique is to provide a simplified model, which is computationally simpler to handle than the original higher order system. Several methods available for reducing the order of a transfer function [1, 2, 3, 4]. All these methods are based on the concept that the dynamical behavior of system is determined by the poles nearest to the imaginary axis, i.e. dominant poles. However, many practical control systems do not have dominant poles; the above methods can not be used in general. Shamash [5] suggested a method for reducing the order of transfer functions using the technique of Pade approximation (P.A). The (P.A) used is equivalent to Taylor series expansion about (s=0). The method gives the correct steady-state response. However, initial transient response may not be good. Also, like many other approximation methods based on power series expansion [6], the stability of the model is not guaranteed, even if the original system is stable. Hutton and Fried land [7] proposed Routh approximation technique, this reduced model preserves high-frequency characteristics and mainly based on the standard Routh-Harwitiz array. This technique deals with the transient response part of the reduced model.

2-Pade Approximation and Modified Routh Array (PAMRA)

This paper presents a combination of Pade Approximation and Modified Routh Array (PAMRA) for model reduction of the transfer function. The proposed method (PAMRA) is a direct method since; it does not involve recursive formulae in deriving the reduced order models. The procedure is purely analytical, simple and needs less computation, when compared with tedious graphical methods. The proposed (PAMRA) is illustrated with several numerical examples. This technique produces good results, i.e. the difference between the time response of original and simplified system is very small and diminishes rapidly.

Given the high-order transfer function G(s) and by long division of the numerator to the denominator gives: -

$$G(s)= C_0 + C_1S + C_2S^2 + \ldots\ldots\ldots\ldots\ldots \qquad (1)$$

For the function G(s) in (1) to be approximated. Let the following Pade approximation (PA) be defined as [8]: -

$$Y(s) \qquad a_0 + a_1S + a_2S^2 + \ldots\ldots\ldots + a_mS^m$$

$$G(s)= \frac{\quad}{R(s)} = \frac{\quad}{b_0 + b_1S + b_2S^2 + \ldots\ldots +b_nS^n} \quad , \qquad m< n \qquad (2)$$

Where (m) is the order of numerator and (n) is the order of denominator. For the first (m+n) terms of (1) and (2) to be equivalent, it becomes apparent that the following set of relations must hold:-

$$\begin{aligned} a_0 &= b_0C_0 \\ a_1 &= b_0C_1+b_1C_0 \\ a_2 &= b_0C_2+b_1C_1+b_2C_0 \\ . \quad & \ldots \quad \ldots \quad \ldots \\ . \quad & \ldots \quad \ldots \quad \ldots \\ a_m &= b_0C_m+b_1C_{m-1}+\ldots\ldots+b_mC_0 \end{aligned} \qquad (3)$$

Or in matrix form:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ a_m \end{pmatrix} = \begin{pmatrix} C_0 & 0 & 0 & \ldots\ldots\ldots & 0 \\ C_1 & C_0 & 0 & \ldots\ldots\ldots & 0 \\ C_2 & C_1 & C_0 & \ldots\ldots\ldots & 0 \\ & & \ldots\ldots\ldots & & 0 \\ & & \ldots\ldots\ldots & & 0 \\ C_m & C_{m-1} & \ldots\ldots & C_1 & C_0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ b_{n-1} \end{pmatrix} \qquad (4)$$

It must be noted that (n=m+1), from equ. (4), the coefficients [a0 , a1 , a2, ….., am] and [b0 , b1 , b2 , ……, bn-1] are unknown, while the matrix of [C0 , C1 , C2 , …Cm] is known from equ. (1). The numerator (a0, a1… am) of the reduced model can be determined latter, after calculating the denominator (b0, b1… bn-1) from the modified Routh array.

Rearrange equ. (2) as:

$$G(s) = \frac{a_{n-1}+ a_{n-2}S^{n-2} + \ldots\ldots\ldots\ldots +S^{n-1}}{b_n +b_{n-1}S +b_{n-2}S^2 +b_{n-3}S^3 +\ldots\ldots+S^n} \qquad (5)$$

Using (RA) and construct a α-table as following:-

$$b^0_{0}= b_n \qquad b^0_{2}= b_{n-2} \qquad b^0_{4}= b_{n-4} \ldots\ldots\ldots\ldots$$

$$b^1_{0}= b_{n-1} \qquad b^1_{2}= b_{n-3} \qquad b_{4}= b_{n-5} \ldots\ldots\ldots\ldots$$

$$\alpha_1 = \frac{b^0_0}{b^1_0} \qquad b^2_{0}= b^0_2 - \alpha_1 b^1_2 \qquad b^2_{2}= b^0_4 - \alpha_1 b^1_4 \qquad b^2_{4}= b^0_6 - \alpha_1 b^1_6$$

$$\alpha_2 = \frac{b^1_0}{b^2_0} \qquad b^3_{0}= b^1_2 - \alpha_2 b^2_2 \qquad b^3_{2}= b^1_4 - \alpha_2 b^2_4 \qquad \ldots\ldots\ldots \qquad (6)$$

$$\alpha_3 = \frac{b^2_0}{b^3_0} \qquad b^4_{0}= b^2_2 - \alpha_3 b^3_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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Denote the denominator Qk(s) of the reduced model from the polynomials are now representing by [9]:

$$Q_k(s) = S^2 Q_{k-2}(s) + \alpha_k Q_{k-1}(s) \qquad (7)$$

With $Q_{-1}(s) = 1/s$, $Q_0 (s) = 1$

For second-order reduced model (k=2) equ. (7) becomes:

$$Q_2(s) = S^2 + \alpha_2 S + \alpha_1\alpha_2$$

$$= S^2 + b_1 S + b_0 \qquad (8)$$

Once the reduced model denominator has been determined, the numerator coefficients (a0 , a1, …….) are obtained from equ.(3)

3- Algorithm

Step 1: by long division of G(s), calculate the coefficients (C0, C1, ……., Cm), equ.(1)

Step 2: Construct the α-table by using modified Routh array, equ. (6)

Step 3: Use equ.(7) to calculate the denominator coefficients of the reduced model

Step 4: Use equ.(3) to calculate the numerator coefficients of the reduced model.

Step 5: Apply the transient response program by using MATLAB package.

4-Numerical examples

In this section the transient response of various reduced models are compared with the full model by way of numerical examples. The computer-oriented algorithm is developed for model reduction method in time domain.

Example 1:

Consider the sixth-order model described by the transfer function [10]:-

$$G(s)= \frac{9 + 23.25\, S + 30.2\, S^2 + 22.25\, S^3 + 9\, S^4 + S^5}{15 + 69.5\, S + 119\, S^2 + 100\, S^3 + 45\, S^4 + 10.5\, S^5 + S^6}$$

For the proposed method, applying the <u>step 1</u> algorithm, we get:-

$$G(s)= 0.6 - 1.23\, S + 2.955\, S^2 - 6.453\, S^3 + \ldots\ldots\ldots$$

$$= C_0 + C_1\, S + C_2\, S^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

<u>Step 2</u>

	15	119	45	1
	69.5	100	10.5	
$\alpha_1 = \frac{15}{69.5} = 0.216$	97.4	42.7		

$$\alpha_2 = \frac{69.5}{97.4} = 0.71$$

69.5 9.8

Step 3

If the reduction order k=2

$$Q_2(s) = S^2 + 0.71\ S + 0.154 \qquad \text{(the denominator coefficients)}$$

$$= S^2 + b_1\ S + b_0$$

Step 4

$$a_0 = b_0 C_0 = (0.154)\ (0.6) = 0.09$$

$$a_1 = b_0 C_1 + b_1 C_0 = (0.154)\ (-1.23) + (0.71)\ (0.6) = 0.24$$

The reduced model R(s) for second-order model reduction is:-

$$R_2(s) = \frac{0.24\ S + 0.09}{S^2 + 0.71\ S + 0.154}$$

While for time-moment matching method by Lees [10] is:-

$$R_{2L}(s) = \frac{0.408\ S + 0.6}{2.73\ S^2 + 3.73\ S + 1}$$

Step 5

To demonstrate the difference between the original system G(s) and the two methods, using the transient response by MATLAB package. Fig (1) shows the transient response of the original system and the reduced model R2(s) and there is very good agreement, but there is a significant difference to the Lees method due to this method depends on the time-moment matching (steady-state part only).

Example 2:

Consider the third-order model described by the transfer function [11] is:-

$$G(s)=\frac{7.5}{7.5 + 26.8\ S + 6.3\ S^2 + S^3}$$

The full model has closed-loop poles at (-0.3, -3+ 4j). The reduced model R(s) obtained b applying the algorithm (step 1-5) for the first-order model reduction is:-

$$R_1(s)=\frac{0.28}{S+ 0.28}$$

While for reduction by Zmood method [11], is:-

$$R_{1Z}(s)=\frac{1.073}{3.33\ S+ 1}$$

It can be seen that there is a very good agreement between G(s) and the proposed method R1(s) but there is a significant difference between G(s) and R1Z(s), see fig (2) Both methods depends on the dominant pole of the full model (S= -0.3) which is considerably closer to the imaginary axis of S-plane than all the remaining complex conjugate pair of poles (S=-3+ 4j).

Example 3:

Consider the Fifth-order model described by the transfer function is:-

$$G(s)=\frac{72+ 99\ S + 11\ S^2 + 7\ S^3 + S^4}{120 + 274\ S + 225\ S^2 + 85\ S^3 + 15\ S^4 + S^5}$$

The reduced model R(s) obtained by applying the algorithm (step 1-5) for second-order and third-order model reduction are:-

0.572 S + 0.3836

$$R_2(s) = \frac{}{S^2 + 1.46\,S + 0.64}$$

$$R_3(s) = \frac{1.233\,S^2 + 1.875\,S + 0.774}{S^3 + 3.38\,S^2 + 4.3\,S + 1.29}$$

It can be seen from fig (3), the reduced model R3(s) is more agreement to the original system compared with R2(s) because there is small losses of information. The order of reduction (k) is increased, the behavior of reduced system approach to the original system behavior.

Example 4:

Consider the fourth-order model described by the transfer function [12] is:-

$$G(s) = \frac{1 + S + 0.5\,S^2}{1 + 2.034\,S + 1.367\,S^2 + 0.367\,S^3 + 0.0334\,S^4}$$

The reduced model R(s) obtained by applying the algorithm (step 1-5) for first-order and second-order model reduction are:-

$$R_1(s) = \frac{0.5}{S + 0.5}$$

$$R_2(s) = \frac{0.827\,S + 0.857}{S^2 + 1.714\,S + 0.857}$$

Fig. (4) shows the transient response of original system and the two reduced systems, also it is apparent R2(s) is more accurate response than R1(s) , where the less dominant poles of the original model are dropped and the dominant poles of R2(s) are retained and this a desirable property.

Example 5:

Consider the third-order model described by the transfer function [13] is:-

$$G(s)=\frac{40 + 13\, S + S^2}{20 + 32\, S + 13\, S^2 + S^3}$$

The reduced model R(s) obtained by applying the algorithm (step 1-5) for first-order and second-order model reduction are:-

$$R_1(s)=\frac{1.25}{S + 0.625}$$

$$R_2(s) = \frac{3.232 + 1.05\, S}{1.616 + 2.585\, S + S^2}$$

While for reduction by Mitra method [13] is:-

$$R_{2M}(s) = \frac{2 + 0.48\, S}{1 + 1.5144\, S + 0.5171\, S^2}$$

Fig (5) shows the transient response of original system and the two reduced systems, again it can be seen the step responses of transfer functions R2(s) and G(s) are very similar, also Mitra method gives good response because it retained the dominants poles of the original system (S= -1 and S= -1.925), but sometimes resulting unstable reduced models for stable full models and vice verse.

5- Conclusions

The transient response can be characterized by a single time constant for first-order systems, and by two parameters (time constants) for second- order systems. For high order systems the number of parameters required to characterize the transient response will multiply rapidly and in general the transient response will become more complex function of time.

Fortunately for many practical systems considerable simplification (model reduction) of the transient response of complex systems can be achieved, and they can be approximated by the responses of equivalent first and second order systems transfer functions. (PAMRA) is mixed Pade-Routh method makes use of stable reduced polynomials for the denominator and takes advantage of computationally convenient schemes of the Pade method for the numerator. (PAMRA) gives good responses of the reduced models in both transient and steady state parts.

The proposed method is simple for computation, can be preserve the dynamic characteristic of the original model satisfactorily and guarantees to have the same zero initial response as the original system.

Although the method is described for single-input single-output system, it is hoped that the method can be extended to multivariable systems as well.

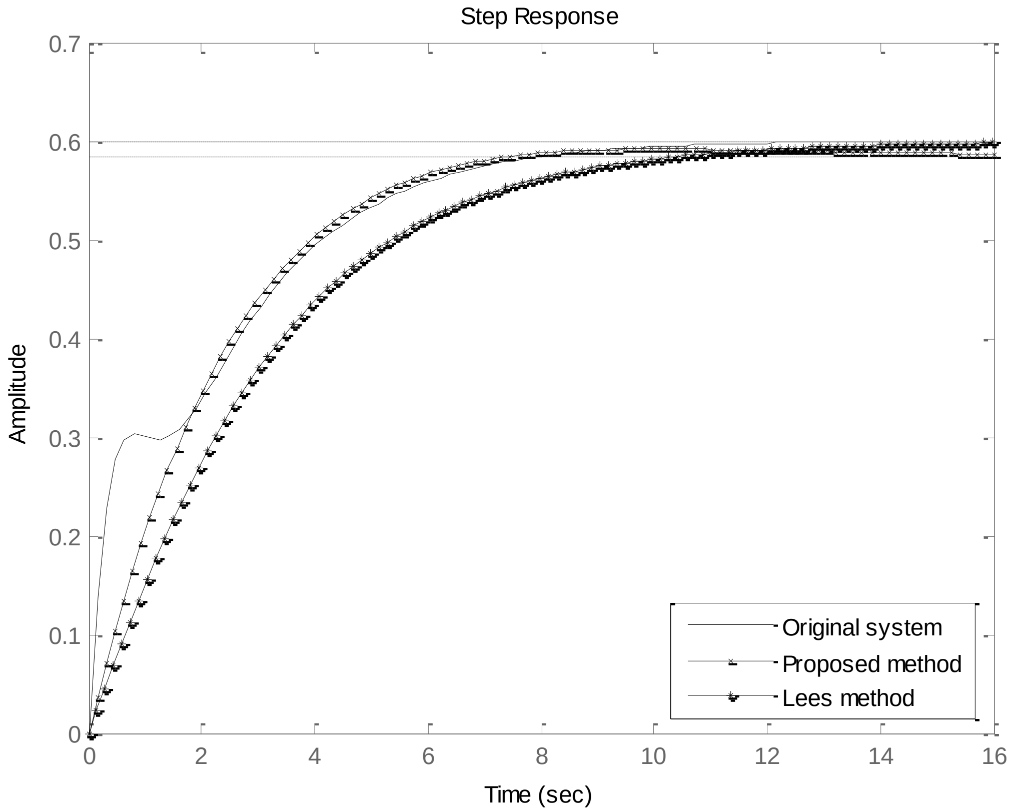

Fig.1: Show the transient response of example 1

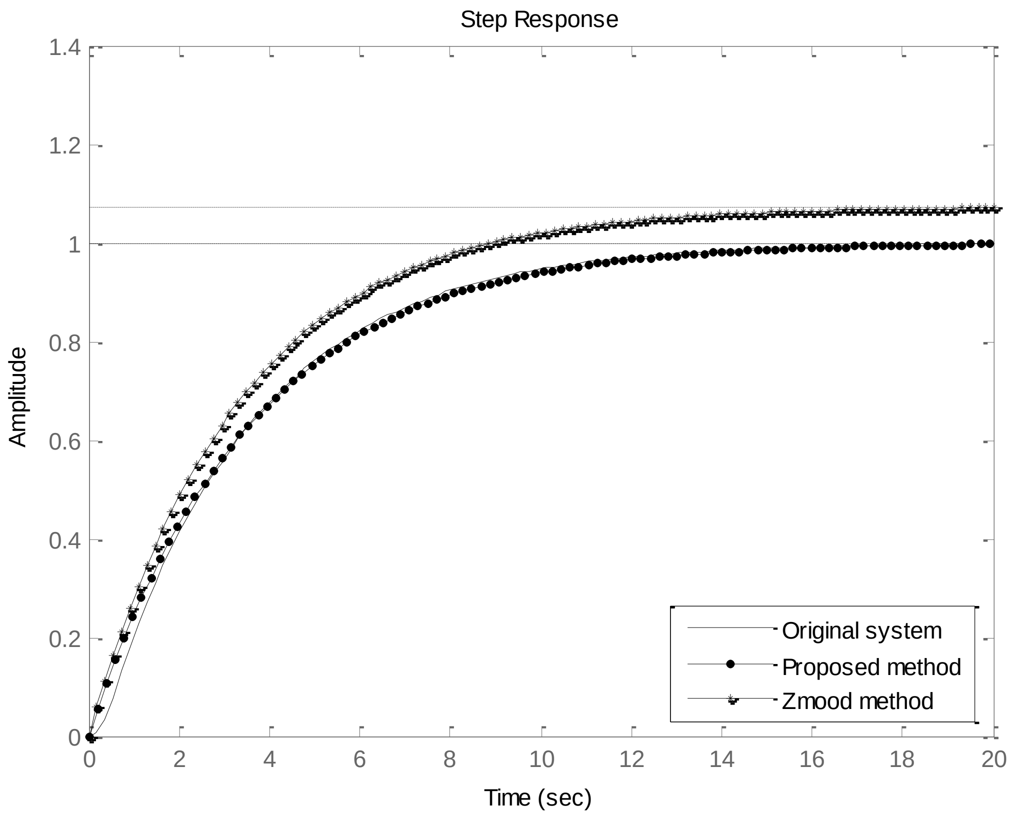

Fig.2: Show the transient response of example 2

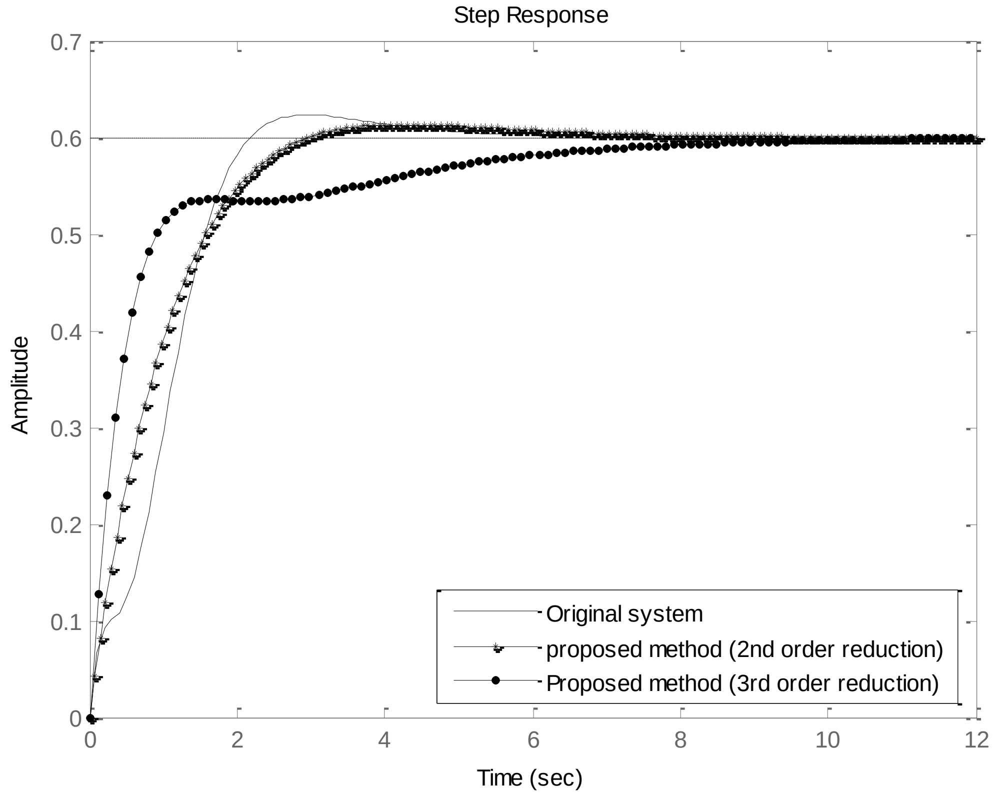

Fig.3: Show the transient response of example 3

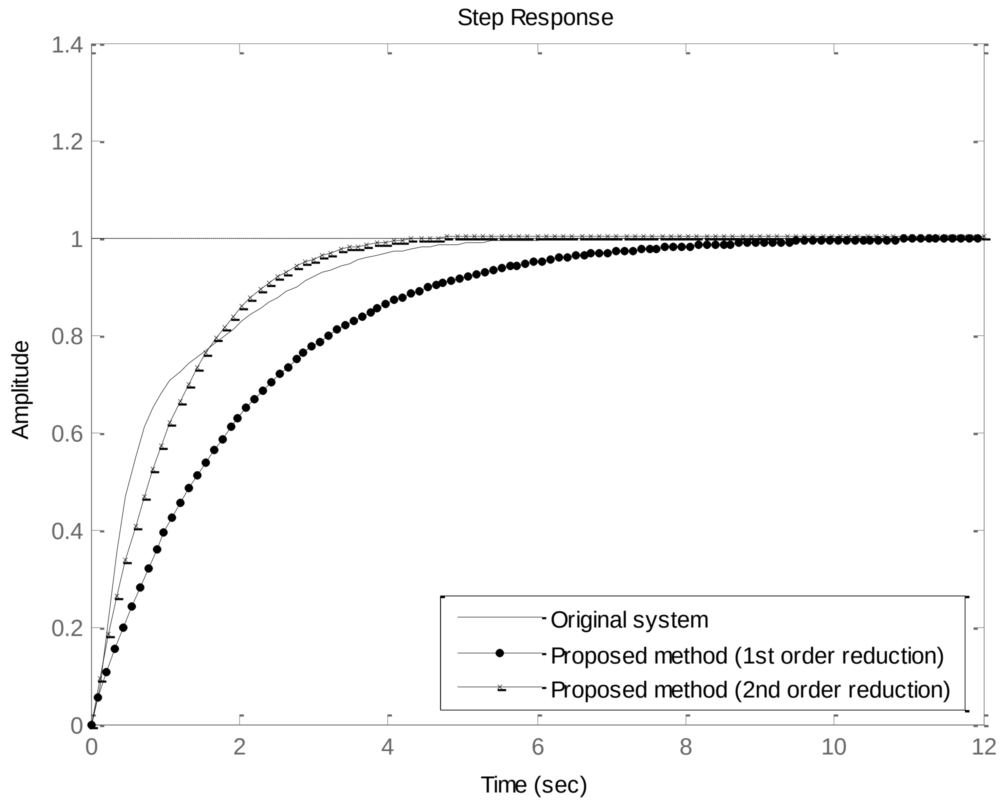

Fig.4: Show the transient response of example 4

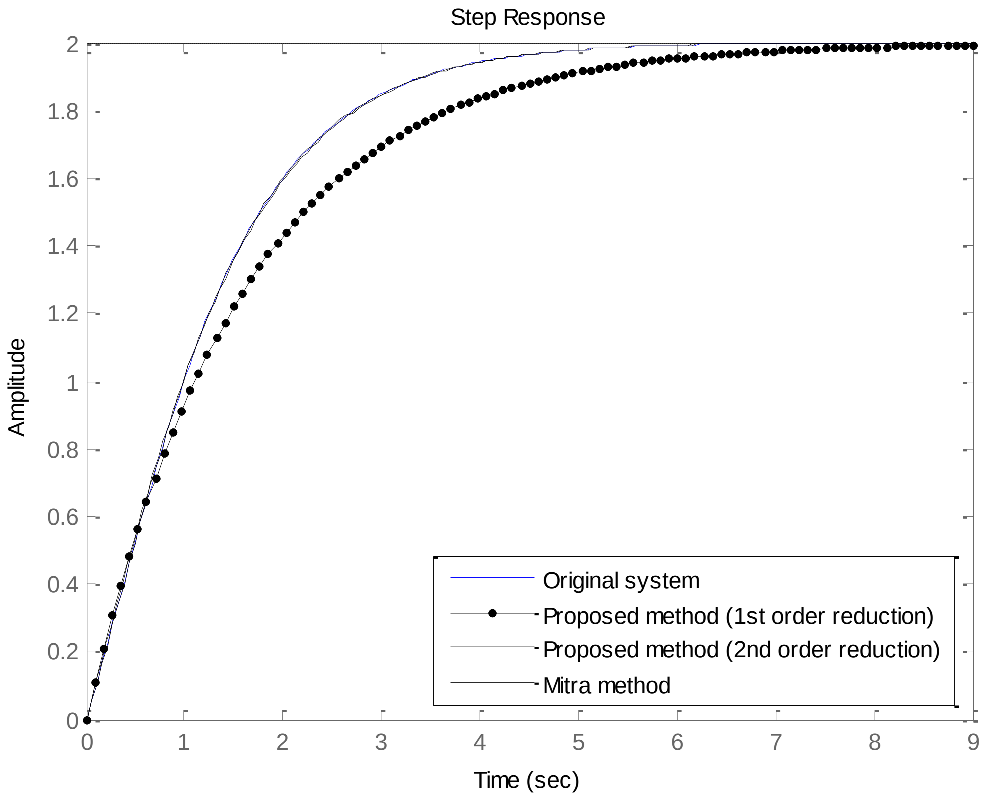

Fig.5: Show the transient response of example 5

6- References

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