Journal of Engineering and Development, Vol. 17, No.5, November 2013 , ISSN 1813- 7822

A NEW FAMILY OF CHAIN FUNCTIONS

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Abstract :

The simulation of the systems required to a mathematic functions, these functions are the arms of the mathematics in the engineering fields and they are used to describe the elements and relations of the systems. This paper proposed a new function annex to the mathematics library to increases the flexibility of the work in the engineering fields; this function is called Dhafer's function. The main structure for Dhafer's function is an empty function called the mother equation, it is continues piecewise with chain form. The mother equation will charged in optional operations to generate a family of functions. These functions are inherent the properties of the mother equation also each element in this family has its special properties. This family of functions has a large amount of advantages over the known functions to day.

The mother equation has three sections the first one is the positive section that is a function to the instantaneous value of the independent variable and the previous value of the function connected by general operator that will call Dhafer operator. The second part is any suitable inverse for the positive section, while the third section is the initial part that is a simple function or constant value that connects the first and the second parts of the function. Some functions in this family have distinct properties such as the continuous factorial and the very fast growing function. The generalization and universe of these equations in the general form called Dhafer's chain function family.

Keywords : Mathematics function, chain function, fast growing, continuous factorial, Zeta functions, pricewise, spline.

عائلة جديدة من الدوال السلسلية

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الخلاصة :

إن محاكاة الأنظمة يتطلب إلى دوال رياضية، وهذه الدوال هي اذرع الرياضيات في المجال الهندسي والتي ستستخدم لوصف العناصر والعلاقات المكونة للأنظمة. إن هذا البحث سيطرح دالة جديدة تضاف غالى مكتبة الرياضيات الغاية منها زيادة مرونة العمل في المجال الهندسي. هذه الدالة ستدعى دالة ظافر، إن الهيكل الأساسي لدالة ظافر هي دالة فارغة تدعى المعادلة الأم وهي دالة قطعية (piecewise) مستمرة بهيكل سلسلي (chain). أن الدالة الأم ستحمل

بعمليات رياضية مناسبة لتوليد عائلة من الدوال. هذه الدوال سوف ترث خواص الدالة الأم مع احتفاظ كل دالة بخواص خاصة بها. هذه العائلة من الدوال ستملك خواص كثيرة لاتمتلكها الدوال المعروفة.

أن معادلة الدالة الأم تملك ثلاث مقاطع الأول هو المقطع الموجب والذي هو دالة تعتمد على قيمة المتغير الحر والقيمة السابقة للدالة وهذه القيم سترتبط بواسطة عملية عامة ستدعى عملية ظافر. المقطع الثاني هو إي ضديد مناسب للمقطع الموجب، بينما يمثل المقطع الثالث هو مقطع الحالة الأولية والذي سيكون دالة بسيطة أو قيمة ثابتة تستطيع ان تصل بين المقطع الأول والثاني. بعض إفراد هذه العائلة تمتلك خواص متميزة مثل المفكوك المستمر والدوال النامية بسرعة. ان مجموعة هذه الدوال والمعادلات ستجمع تحت عنوان عام هو عائلة دوال ظافر السلسلية ***.(Dhafer's chain function family)***

1. Introduction

The function is a way of expressing a special relationship between a value we consider as the input value "x" and an associated output value "y". We write this relationship in the form

$$y = f(x)$$

to indicate that y depends on x. The only constraint on this relationship is that, for every value of x it can get at most one value of y. This is equivalent to the "vertical line property": the graph of a function can intersect a vertical line at most at one point. The set of all allowable x values is called the domain of the function, and the set of all resulting values of y are the range. [1]

Through the simulation of the system (discrete visual model of a system) we find some challenges. These challenges are due to the complexity of the model that has fuzzy, wavelet, AI and a large amount of mathematics functions. An example for the mathematics challenges is the continuous factorial (not Gamma function because it has an infinite values), this problem will solve using approximated expression [2, 3]. Another example is the very fast growing function, that growing faster than n^n function; this problem is solved using the functions iterated exponential [4], Knuth's up-arrow [5], the Ackermann function [6] or the Goodstein's function [7] but some models required to a faster growing and faster attenuation from these functions.

This paper will proposed a new set of functions to solve some mathematics problems and increase the flexibility of the modeling simulation, so it will solve some difficult problem such as the continuous factorial and the fast growing functions. The proposed solutions for these problems fall in three steps first using an equation that has a chain of multiplications that can solve the problem of continuous factorial without any error and easy to use with the simulation language such as PROLOG, the second step is to replace the multiplication operation for this equation by a power operation to solve the second problem and generate an extra fast growing function, the third step is the generalization and universe of these equations in the general form that will called *Dhafer's* chain function family.

This paper has 8 sections, section 2 introduces a brief history, and section 3 shows the definition of the Dhafer's chain function family with form of the mother equation. Section 4 discusses some classic solutions for the mother equation, section 5 presents special properties. While section 6 discuss the additional means for Dhafer operator; section 7 shows the general properties for this functions family, finally section 8 shows some applications for this functions family.

2. Brief History of the Functions

The history of the functions is very old; the early functions are the trigonometric functions were by Egyptians and Babylonians. The ancient Egyptians and Babylonians had known of theorems on the ratios of the sides of similar squares for many centuries. But pre-Hellenic societies lacked the concept of an angle measure and consequently, the sides of triangles were studied instead, a field that would be better called "trilaterometry". [8]

The Babylonian astronomers kept detailed records on the rising and setting of stars, the motion of the planets, and the solar and lunar eclipses, all of which required familiarity with angular distances measured on the celestial sphere. [8]

However, this section will focus on the history of three groups of functions because these groups have an adversary functions in Dhafer's functions family. First group is the graphical functions, second group is the fast growing functions and third group is the continuous factorial functions.

2.1 History of the Graphical Functions

The graphics functions proposed at the time of first trying to plot the 3-D diagrams in computers. Such example for the early works is Zeta functions. Zeta functions of graphs appeared first from the point of view of p-adic groups in the work of Ihara in 1966. He introduced a zeta function of a regular graph and showed that it is a rational function. Then, Bass generalized the Ihara's result on a regular graph to a general graph possibly irregular. In 1999, Bartholdi introduced a zeta function with two variables, which can be regarded as a generalization of the Ihara zeta function, and proved that it is also a rational function. The Ihara type L-function of a graph was developed by Sunada and Hashimoto. The Bartholdi type L-function of a graph was defined by Mizuno and Sato in 2003. So far, many researchers have developed the theory of zeta functions of graphs. [9]

However, thousands of papers and works proved in this field include different ideas and cover different directions. The piecewise functions are the main tool of the 2-D and 3-D graphics. Hence, Yu-Sung Chang [10] is a good example in this field.

This paper will proposed a new group of functions for this field that have a very wide of shapes with simple and uniform expression to enhancement its dimensions and help reduce the required data size and its complexity.

2.2 History of the Fast Growing Functions

The racing of derive and implement a fast growing function was started from centuries but the main stations in this racing are summarize in the following:

1- Iterated exponentials have been puzzling people for centuries, reaching back to mathematicians such as Daniel Bernouli who discussed it with Christian Goldbach back in 1728. Most recently R. Arthur Knoebel, providing a bibliography with no less than 125 entries, has summarized the findings using especially the more recent work and terminology of Daihachiro Sato. Euler, in fact was the first to prove that the function y= f(x)= y $={}_{x}x^{x}$ converges for all x in the interval $[e^{-e}, e^{1/e}]$ and diverges for any other positive value of x. [11]

2- In 1928, Wilhelm Ackermann, a mathematician studying the foundations of computation, originally considered a function A(m, n, p) of three variables, the p-fold iterated exponentiation of m with n, or $m \to n \to p$ as expressed using the Conway chained arrow. [12]

3- The term titration, introduced by Goodstein in his 1947 paper "Transfinite Ordinals in Recursive Number Theory", this work was generalize the recursive base-representation used in Goodstein's theorem to use higher operations, it has gained dominance. It was also popularized in Rudy Rucker's Infinity and the Mind. [13]

4- The term super-exponentiation was published by Bromer in his paper Super-exponentiation in 1987. [4]

2.3 History of the Continuous Factorial Functions

The history of the continuous factorial functions is summarizing in the following:

1- The birth of the real factorial function (1729 - 1826). [13]

2- Euler changed his definition between 1729/30 and 1768. The change is equivalent to a move from x! to Gamma(x). [13]

3- Legendre took on the 1768 definition of Euler, but he did not introduce the transition from x! to Gamma(x). [13]

4- The history and reference for continued fraction formula is: M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions and B. W. Char: "On Stieltjes' continued fraction for the gamma function". [14]

3. Definition of the Mother Dhafer's Function

Dhafer's function is a piecewise continuous chain function connected a set of values that generate from the main independent variable (x) with an optional operator. This function has three sections, first is the positive section that is a function to the instantaneous value of the independent variable and the previous value of the function connected by a general operator that will called Dhafer operator, this section cover the domain from (y, ∞). The second part is any suitable inverse for the positive section that cover the domain from (-∞, -y).

While, the third section is the initial part that is a simple function (or just constant value) that connect the first and the second parts of the function; it is cover the domain from [-y, y].
The main structure for Dhafer's function is an empty function called the mother equation as is in equ. (1).

$$D_y^{\oplus,G,I}(x)=\begin{cases} \frac{x}{y}\oplus D^{\oplus}(x-y) & x>y \\ G(x) & y\geq x\geq -y \\ I\left(D^{\oplus}(x)\right) & x<-y \end{cases} \qquad (1)$$

Where

$\oplus$ is a Dhafer operator it is an optional operator that will gives the main form and properties of the function.

G(x) is the value or function that will grant the continuity for the function.

I(.) is the inverse function of the positive side, hence it depend on the selection of Dhafer operator .

y is the base of the function.

The mother equation will charged in optional operators $\oplus$ to generate a family of functions. This function is not unique function but it is a family of functions that are very flexible, wide range of shapes, complex form. Hence, to show the advantages and applications for this function let at first assume the base y=1as in equ. (2) that will call **unity mother equation** or for more simplicity mother equation for all the next in this paper.

$$D^{\oplus,G,I}(x)=\begin{cases} x\oplus D^{\oplus}(x-1) & x>1 \\ G(x) & 1\geq x\geq -1 \\ I\left(D^{\oplus}(x)\right) & x<-1 \end{cases} \qquad (2)$$

4. Dhafer's Function Solutions

The function solution mainly replace the general operator and functions in equ. (2) by a define operator(s) with suitable functions for them. Generally, it has an infinite solutions that depending on the definition of the Dhafer operator $\oplus$ as first then the definition of the functions G(x) and I(.). This section will discuss some classic solutions (operators) with some of those combinations, also discusses the effect of G(x) in some cases that have more than one suitable function.

This section will show four classic mathematics operations: addition, subtraction, multiplication and the power operators as good examples for the Dhafer's function solutions, the properties of these four functions will discuss in section 5.

4.1 Addition operation

This solution will use the classic addition operation as Dhafer operator $\oplus \rightarrow +$. Hence, the suitable inverse for addition is the negative value that will use for the second section for this solution, while G(x)= x is a suitable connection for the third section as in equ. (3).

$$D^{+}(x)=\begin{cases} x+D^{+}(x-1) & x>1 \\ x & 1\geq x\geq -1 \\ -D^{+}(-x) & x<-1 \end{cases} \qquad (3)$$

4.2 Subtraction operation

This solution will use the classic subtraction operation $\oplus \rightarrow -$ in the first section, the negative value for the second section with G(x)=x as a suitable connection as in equ.(4).

$$D^{-}(x)=\begin{cases} x-D^{-}(x-1) & x>1 \\ x & 1\geq x\geq -1 \\ -D^{-}(-x) & x<-1 \end{cases} \qquad (4)$$

4.3 Multiplication operation

This solution will use the classic multiplication operation $\oplus \rightarrow *$ in the first section, the inverse value for the second section with G(x)=1 as a suitable connection as in equ. (5).

$$D^{*}(x)=\begin{cases} x*D^{*}(x-1) & x>1 \\ 1 & 1\geq x\geq -1 \\ 1/D^{*}(-x) & x<-1 \end{cases} \qquad (5)$$

4.4 Power operation

This solution will use a classic power operation $\oplus \rightarrow \wedge$ with G(x)=1 as shown in equ.(6), it is like equ. (5) with replacing the multiplication operator by power operator.

$$D^{\wedge}(x)=\begin{cases} x^{D^{\wedge}(x-1)} & x>1 \\ 1 & 1\geq x\geq -1 \\ 1/D^{\wedge}(-x) & x<-1 \end{cases} \qquad (6)$$

4.5 Effect of G(x) and I(.) functions

This sub-section discusses the effect of replacing G(x) or/and I(.) functions on the solution of the mother equation. Hence, the positive section will be change in this cases also the change happened in the initial and negative parts as in figure (1).

1- G(x) and I(.) in subtraction operation

This solution will discuss the subtraction operation $\oplus \rightarrow -$ with different initial function G(x) and different I(.) function, that will generate the functions of equations (7a, 7b and 7c) illustrated in figure (1).

$$D^{-,x}(x)=\begin{cases} x-D^{-,x}(x-1) & x>1 \\ x & 1\geq x\geq -1 \\ -D^{-,x}(-x) & x<-1 \end{cases} \quad (7a)$$

$$D^{-,\pm1}(x)=\begin{cases} x-D^{-,\pm1}(x-1) & x>1 \\ \mathrm{sgn}(x) & 1\geq x\geq -1 \\ -D^{-,\pm1}(-x) & x<-1 \end{cases} \quad (7b)$$

$$D^{-,1}(x)=\begin{cases} x-D^{-,1}(x-1) & x>1 \\ 1 & 1\geq x\geq -1 \\ -D^{-,1}(-x) & x<-1 \end{cases} \quad (7c)$$

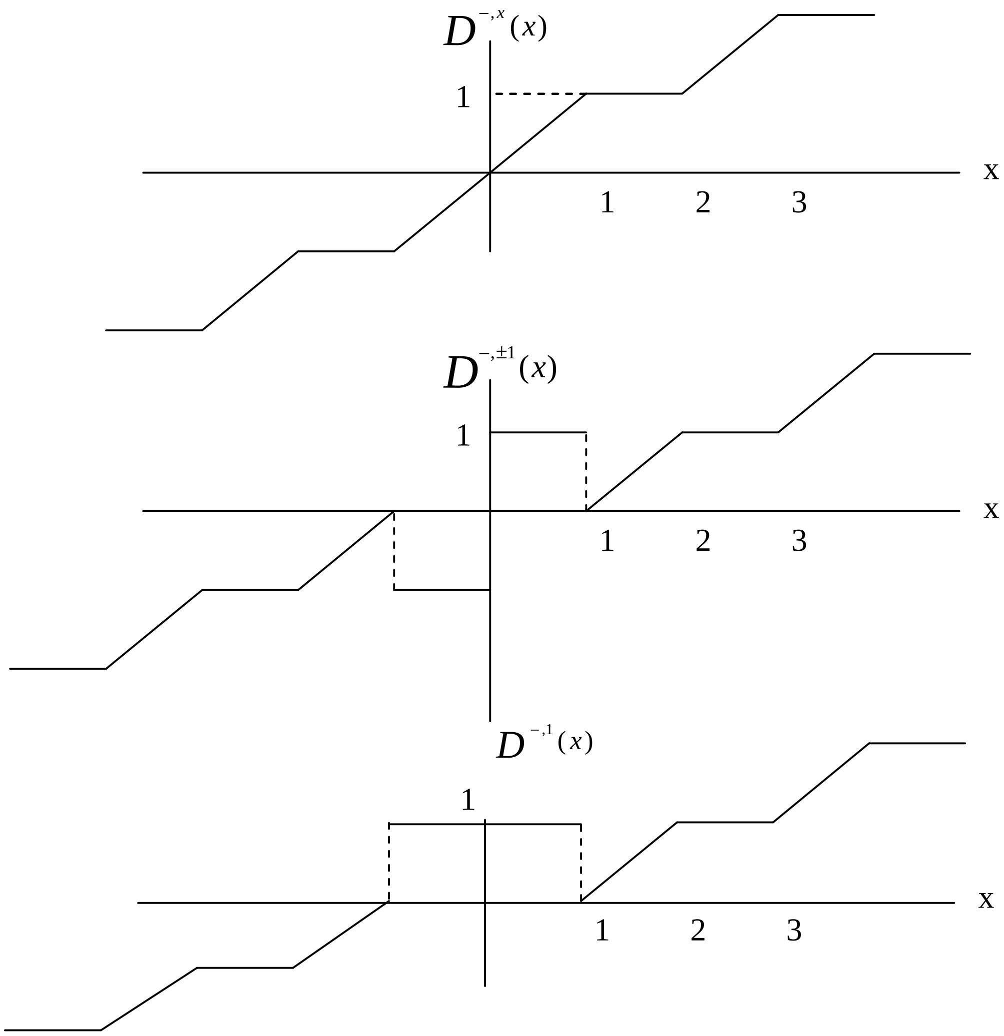

Fig.(1) The effect G(x) and I(.) functions for classic subtraction operation with different cases as in equations (7a, 7b and 7c).

2- G(x) and I(.) in power operation

This solution discusses the classic power operation $\oplus \rightarrow \wedge$ with different initial function G(x) and different I(.) function, that will generate the functions in equations (8a and 8b) (note that the function in equ. (8a) is the same as in equ. (6)).

$$D^{\wedge,1}(x)=\begin{cases} x^{D^{\wedge,1}(x-1)} & x>1 \\ 1 & 1\geq x\geq 0 \\ 1/D^{\wedge,1}(-x) & x<0 \end{cases} \qquad (8a)$$

$$D^{\wedge,x}(x)=\begin{cases} x^{D^{\wedge,x}(x-1)} & x>1 \\ x & 1\geq x\geq 0 \\ -D^{\wedge,x}(-x) & x<0 \end{cases} \qquad (8b)$$

5. Properties of the Classic Dhafer's Functions :

This section discusses the most important mathematics properties for the functions that introduced in section 4.

1. Adder operation properties

The main properties of equ. (3) are listed shown:

i- Derivative: the function has derivatives as in the equations

$$\frac{d}{dx}D^{+}(x)=\begin{cases} \frac{d}{dx}D^{+}(x-1)+1 & x>1 \\ 1 & 1\geq x\geq -1 \\ -\frac{d}{dx}D^{+}(-x+1)+1 & x<-1 \end{cases}=\begin{cases} \lfloor x\rfloor+1 & x>1 \\ 1 & 1\geq x\geq -1 \\ -\lfloor x\rfloor+1 & x<-1 \end{cases}$$

$$\frac{d^2}{dx^2}D^{+}(x)=\begin{cases} d(x) & for\ \mathrm{x}=1,2,\ldots.. \\ 0 & \text{otherwise} \\ d(x) & for\ \mathrm{x}=-1,-2,\ldots.. \end{cases}$$

ii- Integration: the integration of the box function is found in Dhafer's function with adder operator as shown:

$$D^{+}(x)=\int\lfloor x\rfloor dx$$

iii- In addition Dhafer's function with adder operator is equivalent to the square order function where x is integer as in equ. (9).

$$D^{+}(x)=\frac{x^2-1}{2} \quad \text{at x is integer} \qquad (9)$$

2. Subtraction operation properties

The main properties of the subtraction operation of equations (7a, 7b and 7c) are:

i- Derivative: the function has derivatives as in equations

$$\frac{d}{dx}D^{-}(x)=\begin{cases} 1-\frac{d}{dx}D^{-}(x-1) & x>1 \\ 1 & 1\ge x\ge -1 \\ 1-\frac{d}{dx}D^{-}(-x+1) & x<-1 \end{cases}$$

Note that the first derivative is a square wave with period 1 (the period equal to y in general form of mother equation)

$$\frac{d^{2}}{dx^{2}}D^{-}(x)=\begin{cases} d(x) & x=\pm 2,\pm 4,\pm 6,.... \\ -d(x) & x=\pm 1,\pm 3,\pm 5,.... \\ 0 & otherwise \end{cases}$$

ii- Integration: the integration of the square wave function is found in Dhafer's function with subtraction operation as shown:

$$D^{-}(x)=\int\langle\text{square wave with period 1}\rangle dx\,.$$

3. Multiplication operation properties

The main properties of equ. (5) are:

i- Derivative: the function has the following derivatives.

$$\frac{d}{dx}D^{*}(x)=\begin{cases} D^{*}(x-1)+x*\frac{d}{dx}D^{*}(x-1) & x>1 \\ 0 & 1\ge x\ge -1 \\ \left\{-D^{*}(-x+1)+x*\frac{d}{dx}D^{*}(-x+1)\right\}\Big/\left\{D^{*}(-x)\wedge 2\right\} & x<-1 \end{cases}$$

ii- Dhafer's function with multiplications operator is equivalent to the continuous factorial function but it is not equivalent to Gamma function.

Hence, $D^{*}(x)=x!$ where x is integer.

4. Power operation properties

Dhafer's function with power operation is a simple recursive function that produces incredibly large values with simple inputs. The main properties of equ. (6) are (these properties applied on the functions in equations (8a and 8b)).

i- Derivative: the function has derivatives as in equation

$$\frac{d}{dx}D^{\wedge}(x)=\begin{cases} D^{\wedge}(x)\left\{\frac{1}{x}D^{\wedge}(x-1)+\ln(x)*\frac{d}{dx}D^{\wedge}(x-1)\right\} & x>1 \\ 0 & 1\geq x\geq -1 \\ \left\{\frac{1}{x}D^{\wedge}(-x+1)-\ln(-x)*\frac{d}{dx}D^{\wedge}(-x+1)\right\}\Big/\left\{D^{*}(-x)\right\} & x<-1 \end{cases}$$

ii- Dhafer's function with power operation is the most powerful function it is growing faster than n^n.

iii- Dhafer's function with power operation is faster growing than iterated exponential [4], Knuth's up-arrow [5], the Ackermann function [6] and the Goodstein's function [7] as shown in table (1). Hence, it is clear that its values are greater than those of the above mentioned functions for x≥4.

Table (1): values for the famous fast growing functions.

n	Knuth's m/n=1	Ackermann function m/n=1	iterated exponential for ^2n	Goodstein's function	Dhafer's function Power y=1
1	2	3	1	--	1
2	4	4	4	4	2
3	8	5	27	26	9
4	16	6	256	41	262144
5	32	7	3125	60	5^{262144}
6	64	8	46656	83	∞
7	128	9	823543	109	∞
8	256	10	16777216	135	∞
9	512	11	387420489	171	∞
10	1024	12	10000000000	211	∞

∞ is a very large value and not available in ANSI/IEEE Std. 754-1985 system [15].

6. General Dhafer Operator

The mother equation has additional solutions further to the classic mathematic operations mentioned previously. These additional solutions will expand the use of this functions family. These solutions depends on expand the meaning of Dhafer operator to use the non-classic operators. These operators are classified to the following three categories:

1- The non-classic mathematics operation: There are additional mathematics can be use as Dhafer operator such as roots (for positive section $D^{\sqrt{}}(x) = \sqrt[D^{\sqrt{}}(x-1)]{x}$), mod (for positive section $D^{\bmod,x}(x) = x \bmod D^{\bmod,x}(x-1)$), convolution, division and any other mathematics operator.

2- Complex operators: In this case we will use two or more operators such as roots of add ie $\oplus \rightarrow +\sqrt{}$ as the function in equ. (10). However we can use any two or three operators, that will generate any form or function we needs.

$$D^{+\sqrt{}}(x) = \begin{cases} \sqrt{x} + D^{+}(x-1) & x > 1 \\ x & 1 \geq x \geq -1 \\ -D^{+}(-x) & x < -1 \end{cases} \qquad (10)$$

3- The mathematical functions: The additional means for Dhafer operator are the mathematics functions such as sin(.), cos(.) and other mathematical functions like exp(.) and log(.).

7. Properties of General Dhafer's Functions Family

The properties of the Dhafer's functions classified in to two groups; first group is the common properties that apply to all the functions in this family. These properties are derived from the mother equation; this group will called the inherent properties. While, the second group is the specific properties of each function in Dhafer's function family. The main inherent properties for mother equation are:

1- Continues piecewise.

2- Derivative-able, it derivative is a chain functions with general form as in equ. (11).

$$\frac{d}{dx} D^{\oplus}(x) = \begin{cases} f\left(D^{\oplus}(x-1), \frac{d}{dx} D^{\oplus}(x-1)\right) & x > 1 \\ \frac{d}{dx} G(x) & 1 \geq x \geq -1 \\ h\left(D^{\oplus}(-x+1), \frac{d}{dx} D^{\oplus}(-x+1)\right) & x < -1 \end{cases} \qquad (11)$$

Where

$\frac{d}{dx} G(x)$ is a very simple function (generally it is equal to 0 or 1).

$f(.)$ *and* $h(.)$ are two functions depending on $D^{\oplus}(x-1)$ and its derivatives, so it is chain functions.

3- It has negative side.
4- Uniform algorithm, so in result they easy to calculate. This property help us to implement a fast and simple software and hardware systems.
5- Familiar with simulation and AI languages such as PROLOG.
6- Suitable for the engineering applications such as series, polynomials and digital filters (FIR, IIR). So it is easy for fuzzy membership functions and mother wavelets.
7- The output is higher degree than Dhafer operator, for example the operator + generates a power function in order of x^2 as in equ. (9), while the operator * gives a function from the order of X^x. However, the power function will generate a new faster growing function it is faster than the tetration [4].

8. Applications of the Dhafer's Functions in computer since

The Dhafer's functions are available for the general applications of the mathematics functions. They are available for any field but this paper will focus on the applications in the field of the computer since that the functions are born on it. The applications of these functions in the computer since are spread over a wide area. The most important areas are those discussed in the history of the functions for same causes.

8.1 The Computer Graphics

Dhafer's functions have a large amount of functions with uniform formula that helps the designer in 2D or 3D graphics to:

1- Implement a complex forms with small size of data.
2- Reduce the required data memory.
3- Very flexible zooming diagrams.

8.2 The Spline Applications

The word “spline” originated from the term used by ship builders referring to thin wood pieces. It is one of the most important fields in the computer graphics. Over the last 40 years, splines have become very popular in computer graphics, computer animation, and computer-aided design fields. From containers for household goods to state-of-the-art airplanes, it is hard to find any industrial product without spline surfaces. Also, they are widely used in other mathematical studies, such as interpolation and approximation. Dhafer's functions are good alternative functions to the spline functions also they have more flexibility and simpler calculations. [16]

8.3 Fast Growing Applications

The main use for the fast growing functions such as tetration and Dhafer's power function is to represent an extra large (small) numbers that fill out of the physical representations. [17,18] The other use is the simulation for the critical systems (testing stability). Dhafer's power function is better than the other functions mentioned in table (1) because it is faster, easy and lower complexity than the other functions.

8.4 Continuous Factorial

As we known the n! is a discrete, not derive-able and not integrate-able. Therefore, any function or system has factorial function is a discrete, not derive-able and not integrate-able system. Dhafer multiplication function is batter than the other expression because that; these expressions are general approximation so they exact for integer but not granted the real values for fractions so they complex calculations. Generally the continuous factorials are used in different fields such as Laplace transform, Riemann–Liouville integral and Erdélyi–Kober operator.

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